

# **On the structure of continuum and discrete models for elastic response**

Athanasios Tzavaras

University of Crete

Supported by the ACMAC project of European Union FP7

# Outline

Hyperbolic conservation laws

The equations of polyconvex elasticity

Kinetic models for the motion of crystalline surfaces

# Conservation laws

$u : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^n$  satisfies system of conservation laws  $\partial_t u + \operatorname{div} f(u) = 0$   
 $\eta(u) - q(u)$  entropy - entropy flux pair  $\partial_t \eta(u) + \operatorname{div} q(u) = 0$   
sometimes  $\eta(u)$  is convex

entropy pairs generated

$$\nabla^2 \eta(u) \nabla f_\alpha(u) = \nabla f_\alpha(u)^T \nabla^2 \eta(u), \quad \alpha = 1, \dots, d$$

underdetermined  $n = 1$ ; determined  $n = 2, d = 1$ ; overdetermined else

Examples

equations of gas dynamics: conservation of mass, momentum, energy, entropy  
(smooth flows)

equations of isothermal elasticity: conservation of momentum, mechanical energy  
(smooth flows)

$\bar{u} \in W^{1,\infty}$  smooth (conservative) solution  $\partial_t \eta(\bar{u}) + \operatorname{div} q(\bar{u}) = 0$   
 $u \in L^p$  entropy weak solution  $\partial_t \eta(u) + \operatorname{div} q(u) = -\mu \leq 0$

## Approximate solutions - Oscillations

$$\partial_t u^\varepsilon + \operatorname{div} F(u^\varepsilon) = \mathcal{P}_\varepsilon(u^\varepsilon) = \varepsilon \Delta u_\varepsilon$$
$$u^\varepsilon \text{ smooth; } \mathcal{P}_\varepsilon(u) \rightarrow 0 \text{ in } \mathcal{D}'$$

examples: **viscosity**, relaxation, kinetic, variational approximations

$$\partial_t \eta(u^\varepsilon) + \operatorname{div} q(u^\varepsilon) = -\mu_\varepsilon \leq 0$$

implies  $\int \eta(u_\varepsilon) \leq C$

If  $\eta(u) \sim |u|^p$ ,  $p > 1$ , and  $\eta(u) \geq 0$ , introduce the **Young measures**, a parametrized family of probability measures, to describe oscillations

$$f(u^\varepsilon) \rightharpoonup \langle \nu_{x,t}, f(\lambda) \rangle = \int f(\lambda) d\nu_{x,t}(\lambda) \quad \forall \frac{f(\lambda)}{1 + |\lambda|^p} \rightarrow 0$$

For  $\eta(u) \geq 0$  and convex, define **Young measure with concentration**

$$\eta(u^\varepsilon) dx dt \rightharpoonup \langle \nu_{x,t}, \eta(\lambda) \rangle dx dt + \gamma(dx dt)$$

where  $\gamma \in \mathcal{M}^+$ .

# measure valued solutions

DiPerna '85

$$\nu \in \mathcal{P}, u = \langle \nu, \lambda \rangle \in L^p \text{ mv-solution} \quad \partial_t \langle \nu, \lambda \rangle + \operatorname{div} \langle \nu, f(\lambda) \rangle = 0$$

$$\nu \in \mathcal{P}, u \in L^p \text{ dissipative mv-solution}$$

$$\iint \frac{d\theta}{dt} \left( \langle \nu, \eta(\lambda) \rangle dx dt + \gamma(dx dt) \right) + \int \eta(u_0(x)) dx \geq 0 \quad \forall \theta(t) \geq 0 \text{ test function}$$

This is a very-very weak notion of solutions, nevertheless

(a) Let  $u$  be a dissipative mv-solution,  $\bar{u}$  a Lipschitz conservative solution

$$\text{if } u(0) = \bar{u}(0) \quad \text{then } u = \bar{u} \quad a.e.(x, t), \quad \nu = \delta_{\bar{u}(x,t)}$$

Brenier-DeLellis-Szekelyhidi '11, Demoulini-Stuart-T '11

(b) Let  $u$  mv-solution

$$u^h = \frac{1}{h} \int_0^h u(s, \cdot) ds \rightharpoonup u_0 \quad \text{wk in } L^p$$

If  $u$  is a dissipative mv solution, then there is strong-initial trace, i.e. the associated YM  $\kappa_x = \delta_{u_0(x)}$ .

DiPerna '85, DST '11

## Relative entropy

Gibbs principle in thermomechanics: convex entropy provides stability

Relative entropy provides a quantitative measurement of stabilization in thermomechanical theories,

$$\text{relative entropy} \quad \eta(u|\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u})$$

Dafermos 79, DiPerna 79, ...

"Idea of proof for (a)"

$\nu = \nu_{(x,t)}$ ,  $u = \langle \nu, \lambda \rangle$  is entropic mv-solution of system of conservation laws

$$\partial_t u + \text{div} \langle \nu_{(x,t)}, f(\lambda) \rangle = 0$$

$$\partial_t \langle \nu_{(x,t)}, \eta(\lambda) \rangle + \text{div} \langle \nu_{(x,t)}, q(\lambda) \rangle = -\mu_{x,t} \leq 0$$

relative entropy satisfies

$$\partial_t \langle \nu_{x,t}, \eta(\lambda|\bar{u}) \rangle + \text{div} \langle \nu_{x,t}, q(\lambda|\bar{u}) \rangle \leq O(1) \langle \nu_{x,t}, |\lambda - \bar{u}(x,t)|^2 \rangle$$

convexity provides control of the variance  $\int |\lambda - \bar{u}(x,t)|^2 d\nu_{x,t}(\lambda)$

oscillations do not spread.

## Relative entropy in diffusive limits

$u \rightarrow \bar{u}$  in a diffusive limit

$$\begin{array}{ll} u \text{ entropy solution} & \partial_t \eta(u) + \operatorname{div} q(u) = -\mu \leq 0 \\ \bar{u} \text{ smooth entropy solution} & \partial_t \eta(\bar{u}) + \operatorname{div} q(\bar{u}) = -\nu \leq 0 \end{array}$$

Paradigm

$$(\rho, m) \text{ entropy solution of } \begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div} m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div} \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m, \end{cases}$$

$$(\bar{\rho}, \bar{m}) \text{ entropy solution of porous media } \quad \bar{\rho}_t + \frac{1}{\varepsilon} \operatorname{div} \bar{m} = 0, \quad \bar{m} = -\varepsilon \nabla p(\bar{\rho})$$

satisfies entropy dissipation - gradient flow interpretation of porous media **Otto**

$$h(\bar{\rho})_t - \operatorname{div} (h'(\bar{\rho}) \nabla_x p(\bar{\rho})) = -\frac{|\nabla_x p(\bar{\rho})|^2}{\bar{\rho}}.$$

$$\eta(\rho, m) = \frac{1}{2} \frac{|m|^2}{\rho} + h(\rho) \quad h'' = \frac{p'}{\rho}, \quad q(\rho, m) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + mh'(\rho)$$

Define

$$\eta_{rel}(\rho, m) = \eta(\rho, m) - \eta(\bar{\rho}, \bar{m}) - \eta_\rho(\bar{\rho}, \bar{m})(\rho - \bar{\rho}) - \nabla_m \eta(\bar{\rho}, \bar{m}) \cdot (m - \bar{m})$$

but  $\bar{m} = -\varepsilon \nabla p(\bar{\rho})$

$$\eta_{rel}(\rho, m)_t + \frac{1}{\varepsilon} \operatorname{div} q_{rel}(\rho, m) + \frac{1}{\varepsilon^2} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 \leq -Quad - (Linear Error)$$

Convergence from gas dynamics with friction to parabolic equations in the diffusive limit via Lyapunov-type functionals Lattanzio - AT (in preparation)



# The equations of elasticity

$$\frac{\partial^2 y}{\partial t^2} = \operatorname{div} \frac{\partial W}{\partial F}(\nabla y)$$

$$\left\{ \begin{array}{l} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \partial_t v_i = \partial_\alpha \frac{\partial W}{\partial F_{i\alpha}}(F) \\ \partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0 \end{array} \right.$$

motion

$$y(x, t)$$

velocity

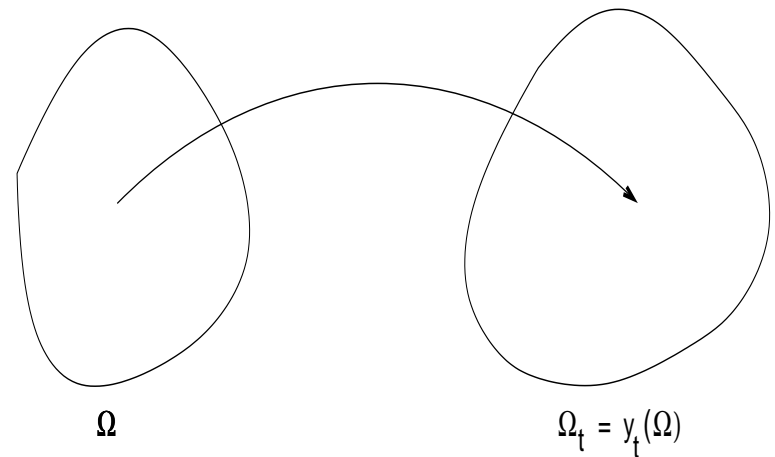
$$v = \frac{\partial y}{\partial t}$$

deformation gradient

$$F = \nabla y$$

$W(F)$  stored energy

$$S = \frac{\partial W}{\partial F}$$



# The equations of elasticity – Requirements

## ■ MATERIAL FRAME INDIFFERENCE

$$W(QF) = W(F) \quad \forall Q \in \mathcal{O}^3$$

## ■ REALIZABILITY OF MECHANICAL MOTIONS

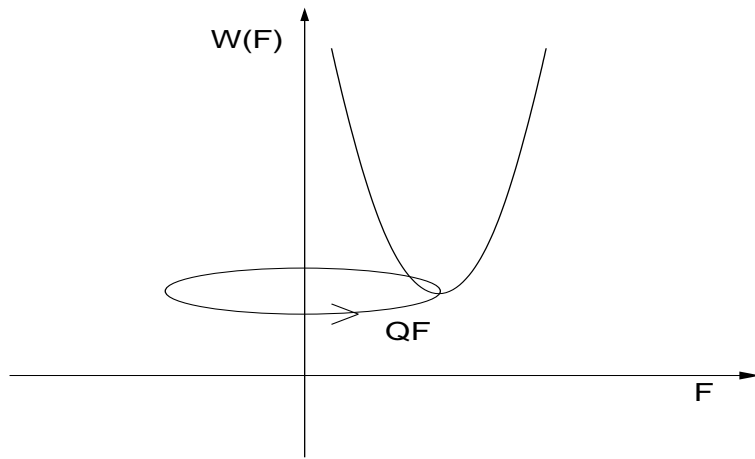
avoid interpenetration of matter

at least positivity of the Jacobian

$$\det F > 0$$

$$W(F) \rightarrow \infty \quad \text{as} \quad \det F \rightarrow 0$$

It is too restrictive to take  $W(F)$  convex



$$\begin{aligned} \text{Hyperbolicity} &\iff \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(F) \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \quad \forall \xi \neq 0, \nu \in \mathcal{S}^2 \\ &\iff W(F) \text{ is rank-1 convex} \end{aligned}$$

Energy identity

$$\partial_t \left( \frac{1}{2} |v|^2 + W(F) \right) + \partial_\alpha \left( v_i \frac{\partial W}{\partial F_{i\alpha}} \right) = 0$$

### QUESTION

Conservation law theory is intricately connected to convexity of the energy.  
What replaces this notion ?

## Notions from elastostatics

$$\min_{y \in W^{1,\infty}} I[y] = \int_{\Omega} W(\nabla y) dx$$

$W(F)$  is **polyconvex**

$$W(F) = g(F, \operatorname{cof} F, \det F) = g \circ \Phi(F) \quad \text{with } g(\Xi) \text{ convex}$$

$\Phi(F)$  is a **null-Lagrangean** iff

$$\int_{\Omega} \Phi(\nabla y + \nabla \phi) dx = \int_{\Omega} \Phi(\nabla y) dx \quad \forall y \in W^{1,p}, \phi \in C_c^{\infty}$$

$$\iff \partial_{\alpha} \left( \frac{\partial \Phi}{\partial F_{i\alpha}}(\nabla y) \right) = 0 \quad \text{in } \mathcal{D}'$$

$$\iff \Phi(F) \text{ is rank-1 affine}$$

$$\iff \Phi(F) = \alpha F + \beta \operatorname{cof} F + c \det F$$

If  $\Phi(\nabla y)$  is null-Lagrangean then it is weakly continuous in  $W^{1,p}$ .

## Transport identities

$$\begin{aligned}\frac{\partial}{\partial t} \det F &= \frac{\partial}{\partial x^\alpha} ((\operatorname{cof} F)_{i\alpha} v_i) \\ \frac{\partial}{\partial t} (\operatorname{cof} F)_{k\gamma} &= \frac{\partial}{\partial x^\alpha} (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta} v_i)\end{aligned}$$

T. Qin 98

connected to null-Lagrangians  $\Phi(F) = (F, \operatorname{cof} F, \det F)$

$$\partial_\alpha \left( \frac{\partial \Phi}{\partial F_{i\alpha}} (\nabla y) \right) = 0 \quad \text{in } \mathcal{D}'$$

## Transport identities

$$\begin{aligned}\partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t \Phi^A(F) &= \partial_\alpha \left( \frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \right) \quad A = 1, \dots, 19\end{aligned}$$

# The augmented elasticity system

Elasticity with transport identities; variables  $(v, F)$

$$\begin{aligned}\partial_t v_i &= \partial_\alpha \left( \frac{\partial g}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Phi(F)^A &= \partial_\alpha \left( \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right).\end{aligned}$$

SYMMETRIZED ELASTICITY SYSTEM; variables  $(v, \Xi)$

$$\begin{aligned}\partial_t v_i &= \partial_\alpha \left( \frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Xi^A &= \partial_\alpha \left( \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right).\end{aligned}$$

The enlarged system is symmetrizable

$$\partial_t \left( \frac{1}{2} |v|^2 + g(\Xi) \right) - \partial_\alpha \left( v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) = 0$$

## Relative entropy for polyconvex elasticity

$(v, F)$  approximate solution

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$

$$\partial_t v_i = \partial_\alpha \frac{\partial}{\partial F_{i\alpha}} (g \circ \Phi(F)) + \varepsilon \Delta v_i$$

$(\bar{v}, \bar{F})$  smooth solution of elasticity

relative entropy  $\eta_{rel} = \frac{1}{2} |v - \bar{v}|^2 + g(\Phi(F) | \Phi(\bar{F}))$

$$\partial_t \eta_{rel} + \operatorname{div} q_{rel} + \varepsilon |\nabla(v - \bar{v})|^2 = O(1) |\Phi(F) - \Phi(\bar{F})|^2 + O(\varepsilon)$$

Convergence to smooth solutions of polyconvex elasticity in the semimetric

$$\int |v - \bar{v}|^2 + g(\Phi(F) | \Phi(\bar{F}))$$

Lattanzio-AT 06

## Variational approximation – 1d

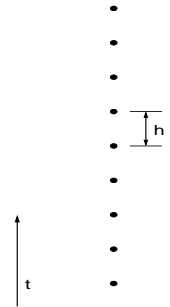
$$\begin{aligned} u_t - v_x &= 0 \\ v_t - \sigma(u)_x &= 0 \end{aligned}$$

time-step discretization: iterates  $(u^j, v^j)$  solve

$$\begin{aligned} \frac{u - u^0}{h} &= v_x \\ \frac{v - v^0}{h} &= \sigma(u)_x \end{aligned}$$

Constructed via the variational problem

$$\min_{\frac{u-u^0}{h}=v_x} \int_I \frac{1}{2} (v - v^0)^2 + W(u) dx$$



- a Scheme emerges from a marching algorithm rather than a target variational problem
- b Scheme produces entropy weak solution for dimension  $d = 1$



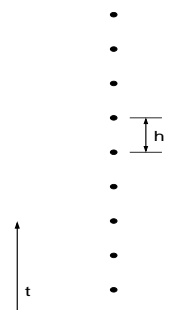
## Variational approximation 3-d

Symmetrized elasticity:

$$\begin{aligned}\partial_t v_i &= \partial_\alpha \left( \frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\ \partial_t \Xi^A &= \partial_\alpha \left( \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right).\end{aligned}$$

suggests the implicit-explicit iterative scheme : time step  $h$

$$\begin{aligned}\frac{v_i^J - v_i^{J-1}}{h} &= \frac{\partial}{\partial x^\alpha} \left( \frac{\partial g}{\partial \Xi^A}(\Xi^J) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) \right) \\ \frac{(\Xi^J - \Xi^{J-1})^A}{h} &= \frac{\partial}{\partial x^\alpha} \left( \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) v_i^J \right).\end{aligned}$$



Iterates  $(v, \Xi)$  constructed by solving the constrained variational problem

Given  $v^0, \Xi^0 = (F^0, Z^0, w^0)$ ,

$$\min \int_{\mathbb{T}^3} \left( \frac{1}{2} |v - v^0|^2 + g(F, Z, w) \right) dx$$

over the affine subspace

$$\mathcal{C} := \left\{ (v, F, Z, w) : \mathbb{T}^3 \rightarrow \mathbb{R}^{22} \text{ subject to the constraints} \right. \\ \left. \begin{aligned} \frac{1}{h} (F_{i\alpha} - F_{i\alpha}^0) &= \partial_\alpha v_i, \\ \frac{1}{h} (Z_{k\gamma} - Z_{k\gamma}^0) &= \partial_\alpha (\epsilon_{ijk} \epsilon_{\alpha\beta\gamma} F_{j\beta}^0 v_i), \\ \frac{1}{h} (w - w^0) &= \partial_\alpha ((\text{cof } F^0)_{i\alpha} v_i) \end{aligned} \right\}.$$

Iterates decrease the mechanical energy, obey bounds

$$\sup_j \int_{\mathbb{T}} 3|v^j|^2 + g(\Xi^j) dx + \sum_j |v^j - v^{j-1}|_{L_x^2}^2 + |\Xi^j - \Xi^{j-1}|_{L_x^2}^2 \leq E_0$$

Under coercivity for  $g$  and bounds for  $g$  and  $\frac{\partial g}{\partial \Xi}$  we have

$$v^h \rightharpoonup v \quad \text{wk in } L^2$$
$$(F^h, Z^h, w^h) \rightharpoonup (F, Z, w) \quad \text{wk in } L^p \times L^q \times L^r$$

and  $(v, F)$  is a measure-valued solution of elasticity, which satisfies the weak form of the geometric transport identities Demoulini-Stuart-AT 01

Uniqueness of classical solutions for polyconvex elasticity within the class of measure-valued solutions Demoulini-Stuart-AT 11

The variational approximation scheme converges whenever the limiting solution is smooth. Miroshnikov-AT 11

## Radial motions of isotropic elastic materials

Radial motions  $y(x, t) = w(R, t) \frac{x}{R}$ ,  $R = |x|$ ,  $x \in \mathbb{R}^3$

$$w_{tt} = \frac{1}{R^2} \partial_R \left( R^2 \frac{\partial \Phi}{\partial v_1} \left( w_R, \frac{w}{R}, \frac{w}{R} \right) \right) - \frac{1}{R} \left( \frac{\partial \Phi}{\partial v_2} + \frac{\partial \Phi}{\partial v_3} \right) \left( w_R, \frac{w}{R}, \frac{w}{R} \right)$$

- $W(F)$  frame indifferent, isotropic, elastic

$$W(F) = \Phi(v_1, v_2, v_2)$$

with  $\Phi$  symmetric function of eigenvalues  $v_1, v_2, v_3$  of  $\sqrt{F^T F}$  and polyconvex, e.g.

$$\Phi = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3) \quad \text{with } h(\delta) \rightarrow +\infty \text{ as } \delta \rightarrow 0+$$

- To represent a physically realizable motion:  $\det F > 0$  with  $F = \nabla y$ .

$$\det F = w_R (w/R)^2 > 0$$

Also sufficient condition for avoiding interpenetration of matter.

Existence and properties for energy minimizers, cavitating solutions

Ball '82, Sivaloganathan '86, ...

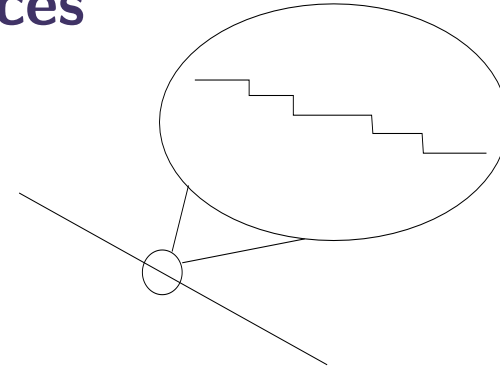
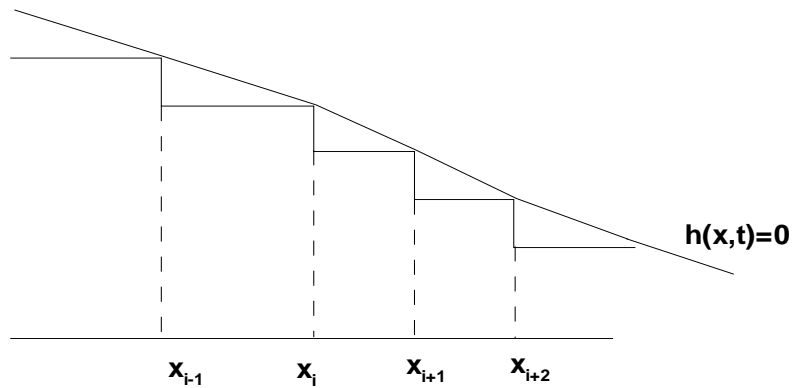
Dynamic, self-similar, cavitating solutions that decrease the total energy

Pericak-Spector '88, ...

Construction of a variational approximation scheme, that preserves the positivity of Jacobians, and is stable and consistent with entropy dissipation.  
(Scheme is based on null-Lagrangians and extensions of the radial elastodynamics system)

Miroshnikov-AT '11

# Mesoscopic theories for crystalline surfaces



(a) Monotonic surface consisting of steps

(b) magnifying lens view

Relaxation of surface to equilibrium below the roughening transition

- Crystalline surface in local equilibrium with its vapor phase. Relaxation effected by adsorption-desorption processes (evaporation-condensation)
- Crystal in vacuum. Relaxation effected through breaking and reconstruction of bonds, process called surface diffusion

## Continuum Thermodynamics models - Spohn - 93

### ■ Evaporation-Condensation

$$\left. \begin{aligned} \frac{\partial h}{\partial t} &= -\kappa(\nabla h) \frac{\delta F}{\delta h} \\ F[h] &= \int dx \sigma(\nabla h) \end{aligned} \right\} \frac{\partial h}{\partial t} = \kappa(\nabla h) \sum_{\alpha} \partial_{x_{\alpha}} \frac{\partial \sigma}{\partial p_{\alpha}}(\nabla h)$$

### ■ Surface Diffusion

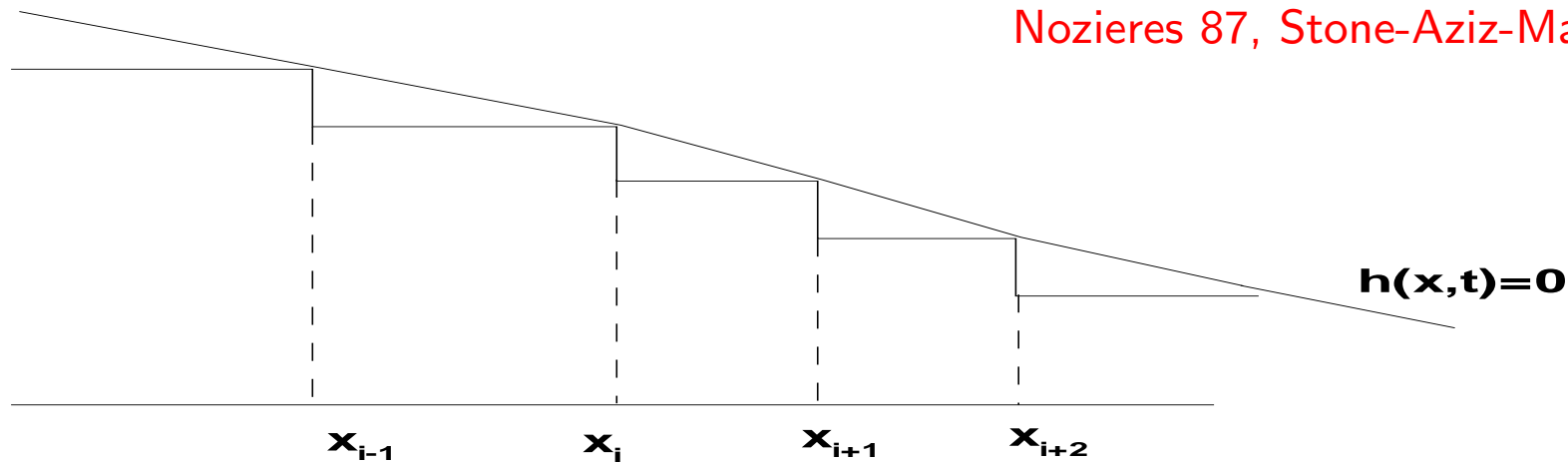
$$\left. \begin{aligned} \frac{\partial h}{\partial t} + \nabla \cdot j &= 0 \\ j &= -\kappa(\nabla h) \nabla \frac{\delta F}{\delta h} \\ F[h] &= \int dx \sigma(\nabla h) \end{aligned} \right\} \frac{\partial h}{\partial t} = -\nabla \cdot \kappa(\nabla h) \nabla \sum_{\alpha} \partial_{x_{\alpha}} \frac{\partial \sigma}{\partial p_{\alpha}}(\nabla h)$$

Mesoscopic step models - Burton-Cabrera-Franck theory - 50's for surface diffusion

# mesoscopic theories vs continuum models

- Issue: Comparison between macroscopic theories and continuum thermodynamic models

Nozieres 87, Stone-Aziz-Margolis 05, ...



$$\dot{x}_i = -\frac{1}{\alpha} \left( g\left(\frac{x_{i+1} - x_i}{\alpha}\right) - g\left(\frac{x_i - x_{i-1}}{\alpha}\right) \right)$$

$$\dot{x}_i = -\frac{h_t}{h_x}$$

$$m_i = \frac{a}{x_{i+1} - x_i} \sim -h_x(x_i, t)$$

as  $a \rightarrow 0$

$$-\frac{h_t}{h_x} = -\frac{1}{m} \partial_x \left[ g\left(\frac{1}{m}\right) \right]$$

$$\partial_t m = \partial_{xx} g\left(\frac{1}{m}\right)$$



# Kinetic Description of ODE System

## Evaporation-Condensation

$$\dot{x}_i = -\frac{1}{\alpha} \left[ g\left(\frac{x_{i+1} - x_i}{\alpha}\right) - g\left(\frac{x_i - x_{i-1}}{\alpha}\right) \right]$$

$$x_1 < \dots < x_{i-1} < x_i < x_{i+1} < \dots < x_N \quad aN = 1$$

Objectives: 1/. Develop a kinetic description

$$F_1^a(y, t) = a \sum_{i=1}^N \delta(y - x_i(t))$$

$$F_2^a(y_1, y_2, t) = a \sum_{i=1}^N \delta(y_1 - x_i(t)) \delta(y_2 - x_{i+1}(t))$$

$$F_n^a(y_1, \dots, y_n, t) = a \sum_{i=1}^N \delta(y_1 - x_i(t)) \delta(y_2 - x_{i+1}(t)) \dots \delta(y_n - x_{i+n-1}(t))$$

# BBGKY Hierarchy

$$F_1^a(x, t) = a \sum_{i=1}^N \delta(x - x_i(t))$$

$$F_n^a(y_1, \dots, y_n, t) = a \sum_{i=1}^N \delta(y_1 - x_i(t)) \delta(y_2 - x_{i+1}(t)) \dots \delta(y_n - x_{i+n-1}(t))$$

The hierarchy for  $F_n^a$  is described by

$$n = 1 : \quad \partial_t F_1^a(x) + \partial_x \left\{ \int dz dy \frac{1}{a} \left[ g\left(\frac{x-z}{a}\right) - g\left(\frac{y-x}{a}\right) \right] F_3^a(z, x, y) \right\} = 0, \quad (1)$$

$$\begin{aligned} n = 2 : \quad \partial_t F_2^a(x, y) + \partial_x \left\{ \int dz \frac{1}{a} \left[ g\left(\frac{x-z}{a}\right) - g\left(\frac{y-x}{a}\right) \right] F_3^a(z, x, y) \right\} \\ + \partial_y \left\{ \int dz \frac{1}{a} \left[ g\left(\frac{y-x}{a}\right) - g\left(\frac{z-y}{a}\right) \right] F_3^a(x, y, z) \right\} = 0, \quad (2) \end{aligned}$$

$$\begin{aligned} n \geq 3 : \quad \partial_t F_n^a(\vec{y}) + \partial_{y_1} \left\{ \int dp \frac{1}{a} \left[ g\left(\frac{y_1-p}{a}\right) - g\left(\frac{y_2-y_1}{a}\right) \right] F_{n+1}^a(p, \vec{y}) \right\} \\ + \sum_{j=2}^{n-1} \partial_{y_j} \left\{ \frac{1}{a} \left[ g\left(\frac{y_j-y_{j-1}}{a}\right) - g\left(\frac{y_{j+1}-y_j}{a}\right) \right] F_n^a(\vec{y}) \right\} \\ + \partial_{y_n} \left\{ \int dp \frac{1}{a} \left[ g\left(\frac{y_n-y_{n-1}}{a}\right) - g\left(\frac{p-y_n}{a}\right) \right] F_{n+1}^a(\vec{y}, p) \right\} = 0. \quad (3) \end{aligned}$$

# Macroscopic Limit

2/ Macroscopic description:  $a \rightarrow 0$ ,  $aN = 1$

Thm Suppose that

$$\zeta_i(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}; a) = v(x_i) + o(1)$$

uniformly on  $i$ , as  $a \rightarrow 0$ ,  $N \rightarrow \infty$ ,  $Na = 1$ .

Then  $F_n^a \rightharpoonup F_n$  and  $F_n$  satisfies

$$\partial_t F_1 + \partial_y (v(y) F_1) = 0$$

$$\partial_t F_n(y_1, \dots, y_n, t) + \sum_{j=1}^n \partial_{y_j} \left( v(y_j) F_n(y_1, \dots, y_n, t) \right) = 0$$

Margetis-AT '09

example: Evaporation-Condensation.  $v = \frac{1}{F_1} \partial_y \left( g \circ \frac{1}{F_1} \right)$

$$\partial_t F_1 + \partial_y \left( \frac{\partial}{\partial y} \left( g \circ \frac{1}{F_1} \right) \right) = 0$$

### 3/ Correlation functions

$$\begin{aligned}\partial_t F_1(y, t) + \partial_y (v(y, t) F_1) &= 0 \\ \partial_t F_2(x, y, t) + \partial_y (v(x, t) F_2) + \partial_z (v(y, t) F_2) &= 0\end{aligned}$$

High order correlation functions are expressed in terms of  $F_1$  via the formulas

$$\begin{aligned}F_2(x, y, t) &= F_1(x, t) F_1(y, t) C(h(x, t), h(y, t)) \\ F_n(y_1, \dots, y_n, t) &= F_1(y_1, t) \dots F_1(y_n, t) C(h(y_1, t), \dots, h(y_n, t))\end{aligned}$$

where  $h_x = -F_1$  and the shape  $C$  captures the correlation of the initial data.

Formula describes the evolution of correlation functions  
contrast to propagation of chaos