On the structure of continuum and discrete models for elastic response

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Outline

Hyperbolic conservation laws
The equations of polyconvex elasticity
Kinetic models for the motion of crystalline surfaces

Conservation laws

$u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^n$ satisfies system of conservation laws $\quad \partial_t u + \text{div}\, f(u) = 0$

$\eta(u) - q(u)$ entropy - entropy flux pair $\quad \partial_t \eta(u) + \text{div}\, q(u) = 0$

sometimes $\eta(u)$ is convex

entropy pairs generated

$$\nabla^2 \eta(u) \nabla f_\alpha(u) = \nabla f_\alpha(u)^T \nabla^2 \eta(u), \quad \alpha = 1, ..., d$$

underdetermined $n = 1$; determined $n = 2$, $d = 1$; overdetermined else

Examples

equations of gas dynamics: conservation of mass, momentum, energy, entropy (smooth flows)

equations of isothermal elasticity: conservation of momentum, mechanical energy (smooth flows)

$\bar{u} \in W^{1,\infty}$ smooth (conservative) solution $\quad \partial_t \eta(\bar{u}) + \text{div}\, q(\bar{u}) = 0$

$u \in L^p$ entropy weak solution $\quad \partial_t \eta(u) + \text{div}\, q(u) = -\mu \leq 0$

Approximate solutions - Oscillations

$$\partial_t u^\varepsilon + \operatorname{div} F(u^\varepsilon) = \mathcal{P}_\varepsilon(u^\varepsilon) \; = \varepsilon \triangle u_\varepsilon$$

$$u^\varepsilon \text{ smooth;} \quad \mathcal{P}_\varepsilon(u) \to 0 \;\text{ in } \mathcal{D}'$$

examples: viscosity, relaxation, kinetic, variational approximations

$$\partial_t \eta(u^\varepsilon) + \operatorname{div} q(u^\varepsilon) = -\mu_\varepsilon \le 0$$

$$\text{implies} \quad \int \eta(u_\varepsilon) \le C$$

If $\eta(u) \sim |u|^p$, $p > 1$, and $\eta(u) \ge 0$, introduce the Young measures, a parametrized family of probability measures, to describe oscillations

$$f(u^\varepsilon) \rightharpoonup \langle \nu_{x,t}, f(\lambda) \rangle = \int f(\lambda) d\nu_{x,t}(\lambda) \qquad \forall \frac{f(\lambda)}{1 + |\lambda|^p} \to 0$$

For $\eta(u) \ge 0$ and convex, define Young measure with concentration

$$\eta(u^\varepsilon) dx\, dt \rightharpoonup \langle \nu_{x,t}, \eta(\lambda) \rangle dx\, dt + \gamma(dx\, dt)$$

where $\gamma \in \mathcal{M}^+$.

measure valued solutions

DiPerna '85

$$\nu \in \mathcal{P},\ u = <\nu, \lambda> \in L^p \text{ mv-solution} \qquad \partial_t \langle \nu, \lambda \rangle + \text{div}\, \langle \nu, f(\lambda) \rangle = 0$$

$$\nu \in \mathcal{P},\ u \in L^p \text{ dissipative mv-solution}$$

$$\iint \frac{d\theta}{dt} \Big(\langle \nu, \eta(\lambda) \rangle dx dt + \gamma(dx\, dt) \Big) + \int \eta(u_0(x)) dx \geq 0 \qquad \forall \theta(t) \geq 0 \text{ test function}$$

This is a very-very weak notion of solutions, nevertheless

(a) Let u be a dissipative mv-solution, $\bar{u}$ a Lipschitz conservative solution

$$\text{if } u(0) = \bar{u}(0) \quad \text{then} \quad u = \bar{u} \quad a.e.(x,t), \quad \nu = \delta_{\bar{u}(x,t)}$$

Brenier-DeLellis-Szekelyhidi '11, Demoulini-Stuart-T '11

(b) Let u mv-solution

$$u^h = \frac{1}{h} \int_0^h u(s, \cdot) ds \rightharpoonup u_0 \quad \text{wk in } L^p$$

If u is a dissipative mv solution, then there is strong-initial trace, i.e. the associated YM $\kappa_x = \delta_{u_0(x)}$.

DiPerna '85, DST '11

Relative entropy

Gibbs principle in thermomechanics: convex entropy provides stability
Relative entropy provides a quantitative measurement of stabilization in thermomechanical theories,

relative entropy $\qquad \eta(u|\bar{u}) = \eta(u) - \eta(\bar{u}) - \nabla\eta(\bar{u})(u - \bar{u})$

Dafermos 79, DiPerna 79, ...

"Idea of proof for (a)"
$\nu = \nu_{(x,t)}$, $u =< \nu, \lambda >$ is entropic mv-solution of system of conservation laws

$$\partial_t u + \text{div } < \nu_{(x,t)}, f(\lambda) > = 0$$
$$\partial_t < \nu_{(x,t)}, \eta(\lambda) > + \text{div } < \nu_{(x,t)}, q(\lambda) > = -\mu_{x,t} \leq 0$$

relative entropy satisfies

$$\partial_t < \nu_{x,t}, \eta(\lambda|\bar{u}) > + \text{ div } < \nu_{x,t}, q(\lambda|\bar{u}) > \leq \ O(1) < \nu_{x,t}, |\lambda - \bar{u}(x,t)|^2 >$$

convexity provides control of the variance $\int |\lambda - \bar{u}(x,t)|^2 d\nu_{x,t}(\lambda)$
oscillations do not spread.

Relative entropy in diffusive limits

$u \to \bar{u}$ in a diffusive limit

$$\begin{aligned} u \text{ entropy solution} \qquad & \partial_t \eta(u) + \operatorname{div} q(u) = -\mu \le 0 \\ \bar{u} \text{ smooth entropy solution} \qquad & \partial_t \eta(\bar{u}) + \operatorname{div} q(\bar{u}) = -\nu \le 0 \end{aligned}$$

Paradigm

$$(\rho, m) \text{ entropy solution of} \qquad \begin{cases} \rho_t + \dfrac{1}{\varepsilon} \operatorname{div} m = 0 \\ \\ m_t + \dfrac{1}{\varepsilon} \operatorname{div} \dfrac{m \otimes m}{\rho} + \dfrac{1}{\varepsilon} \nabla_x p(\rho) = -\dfrac{1}{\varepsilon^2} m, \end{cases}$$

$$(\bar{\rho}, \bar{m}) \text{ entropy solution of porous media} \qquad \bar{\rho}_t + \frac{1}{\varepsilon} \operatorname{div} \bar{m} = 0, \quad \bar{m} = -\varepsilon \nabla p(\bar{\rho})$$

satisfies entropy dissipation - gradient flow interpretation of porous media Otto

$$h(\bar{\rho})_t - \operatorname{div} \big(h'(\bar{\rho}) \nabla_x p(\bar{\rho}) \big) = -\frac{|\nabla_x p(\bar{\rho})|^2}{\bar{\rho}}.$$

$$\eta(\rho,m)=\frac{1}{2}\frac{|m|^2}{\rho}+h(\rho) \qquad h''=\frac{p'}{\rho}\,, \qquad q(\rho,m)=\frac{1}{2}m\frac{|m|^2}{\rho^2}+mh'(\rho)$$

Define

$$\eta_{rel}(\rho,m)=\eta(\rho,m)-\eta(\bar\rho,\bar m)-\eta_\rho(\bar\rho,\bar m)(\rho-\bar\rho)-\nabla_m\eta(\bar\rho,\bar m)\cdot(m-\bar m)$$

but $\bar m=-\varepsilon\nabla p(\bar\rho)$

$$\eta_{rel}(\rho,m)_t+\frac{1}{\varepsilon}\mathrm{div}\,q_{rel}(\rho,m)+\frac{1}{\varepsilon^2}\rho\left|\frac{m}{\rho}-\frac{\bar m}{\bar\rho}\right|^2\le -Quad-(LinearError)$$

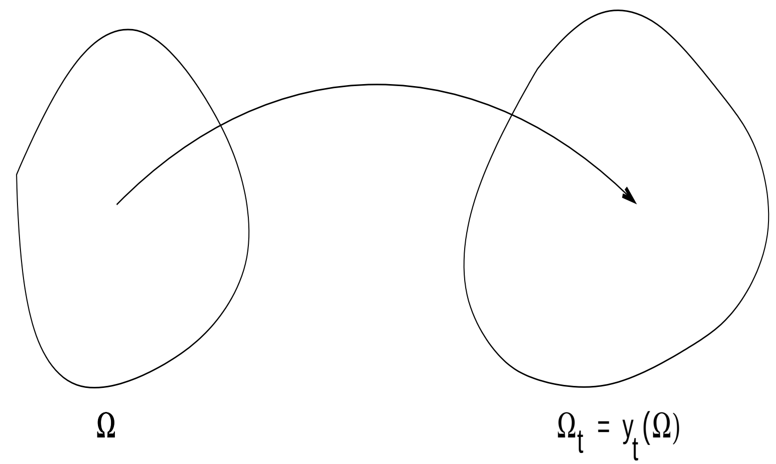
Convergence from gas dynamics with friction to parabolic equations in the diffusive limit via Lyapunov-type functionals Lattanzio - AT (in preparation)

The equations of elasticity

$$\frac{\partial^2 y}{\partial t^2} = \mathrm{div}\,\frac{\partial W}{\partial F}(\nabla y)$$

$$\begin{cases} \partial_t F_{i\alpha} = \partial_\alpha v_i \\ \partial_t v_i = \partial_\alpha \dfrac{\partial W}{\partial F_{i\alpha}}(F) \\ \partial_\alpha F_{i\beta} - \partial_\beta F_{i\alpha} = 0 \end{cases}$$

motion	$y(x,t)$
velocity	$v = \dfrac{\partial y}{\partial t}$
deformation gradient	$F = \nabla y$
$W(F)$ stored energy	$S = \dfrac{\partial W}{\partial F}$



The equations of elasticity – Requirements

- MATERIAL FRAME INDIFFERENCE

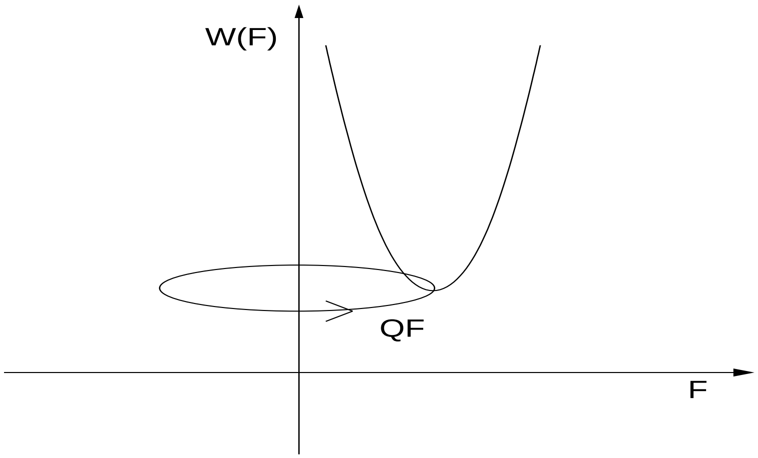
$$W(QF) = QW(F) \quad \forall\, Q \in \mathcal{O}^3$$

- REALIZIBILITY OF MECHANICAL MOTIONS

  avoid interpenetration of matter

  at least positivity of the Jacobian

$$\det F > 0$$



$$W(F) \to \infty \quad \text{as} \quad \det F \to 0$$

It is too restrictive to take $W(F)$ convex

$$\text{Hyperbolicity} \quad \iff \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(F)\, \xi_i \xi_j \nu_\alpha \nu_\beta > 0 \quad \forall \xi \neq 0\,,\ \nu \in \mathcal{S}^2$$

$$\iff \quad W(F) \quad \text{is rank-1 convex}$$

Energy identity

$$\partial_t \left(\frac{1}{2} |v|^2 + W(F) \right) + \partial_\alpha \left(v_i \frac{\partial W}{\partial F_{i\alpha}} \right) = 0$$

Question

Conservation law theory is intricately connected to convexity of the energy. What replaces this notion ?

Notions from elastostatics

$$\min_{y \in W^{1,\infty}} I[y] = \int_\Omega W(\nabla y)\, dx$$

$W(F)$ is polyconvex

$$W(F) = g(F, \mathrm{cof}\, F, \det F) = g \circ \Phi(F) \qquad \text{with } g(\Xi) \text{ convex}$$

$\Phi(F)$ is a null-Lagrangean iff

$$\begin{aligned}
&\int_\Omega \Phi(\nabla y + \nabla \phi)\, dx = \int_\Omega \Phi(\nabla y) dx \quad \forall\, y \in W^{1,p},\ \phi \in C_c^\infty \\
\iff \quad & \partial_\alpha \left(\frac{\partial \Phi}{\partial F_{i\alpha}} (\nabla y) \right) = 0 \quad \text{in } \mathcal{D}' \\
\iff \quad & \Phi(F) \text{ is rank-1 affine} \\
\iff \quad & \Phi(F) = \alpha\, F + \beta\, \mathrm{cof}\, F + c \det F
\end{aligned}$$

If $\Phi(\nabla y)$ is null-Lagrangean then it is weakly continuous in $W^{1,p}$.

J. Ball 77, J. Ericksen 62

Transport identities

$$\frac{\partial}{\partial t}\mathrm{det}\, F = \frac{\partial}{\partial x^\alpha}\big((\mathrm{cof}\, F)_{i\alpha} v_i\big)$$
$$\frac{\partial}{\partial t}(\mathrm{cof}\, F)_{k\gamma} = \frac{\partial}{\partial x^\alpha}(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}v_i)$$

T. Qin 98

connected to null-Lagrangians $\Phi(F) = (F, \mathrm{cof}\, F, \det F)$

$$\partial_\alpha\left(\frac{\partial\Phi}{\partial F_{i\alpha}}(\nabla y)\right) = 0 \quad \text{in } \mathcal{D}'$$

Transport identities

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t \Phi^A(F) = \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}\, v_i\right) \quad A = 1, ..., 19$$

The augmented elasticity system

Elasticity with transport identities; variables (v, F)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Phi(F)^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

Symmetrized elasticity system; variables (v, Ξ)

$$\partial_t v_i = \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right)$$
$$\partial_t \Xi^A = \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).$$

The enlarged system is symmetrizable

$$\partial_t \left(\frac{1}{2}|v|^2 + g(\Xi) \right) - \partial_\alpha \left(v_i \frac{\partial g(\Xi)}{\partial \Xi^A} \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} \right) = 0$$

Relative entropy for polyconvex elasticity

(v, F) approximate solution

$$\partial_t F_{i\alpha} = \partial_\alpha v_i$$
$$\partial_t v_i = \partial_\alpha \frac{\partial}{\partial F_{i\alpha}} (g \circ \Phi(F)) + \varepsilon \triangle v_i$$

$(\bar{v}, \bar{F})$ smooth solution of elasticity

relative entropy $\quad \eta_{rel} = \frac{1}{2}|v - \bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$

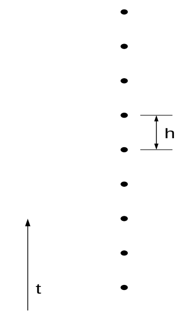
$$\partial_t \eta_{rel} + \operatorname{div} q_{rel} + \varepsilon|\nabla(v - \bar{v})|^2 = O(1)|\Phi(F) - \Phi(\bar{F})|^2 + O(\varepsilon)$$

Convergence to smooth solutions of polyconvex elasticity in the semimetric

$$\int |v - \bar{v}|^2 + g\left(\Phi(F)|\Phi(\bar{F})\right)$$

Lattanzio-AT 06

Variational approximation – 1d



$$
\begin{aligned}
u_t - v_x &= 0 \\
v_t - \sigma(u)_x &= 0
\end{aligned}
$$

time-step discretization: iterates (u^j, v^j) solve

$$
\begin{aligned}
\frac{u - u^0}{h} &= v_x \\
\frac{v - v^0}{h} &= \sigma(u)_x
\end{aligned}
$$

Constructed via the variational problem

$$
\min_{\frac{u-u^0}{h} = v_x} \int_I \frac{1}{2}(v - v^0)^2 + W(u)\, dx
$$

a Scheme emerges from a marching algorithm
rather than a target variational problem

b Scheme produces entropy weak solution for dimension $d = 1$

Demoulini-Stuart-T. '00

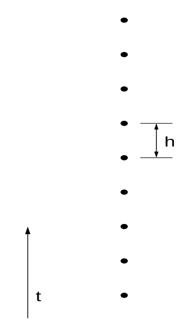
Variational approximation 3-d

Symmetrized elasticity:

$$
\begin{aligned}
\partial_t v_i &= \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\, v_i \right).
\end{aligned}
$$

suggests the implicit-explicit iterative scheme : time step h

$$
\begin{aligned}
\frac{v_i^J - v_i^{J-1}}{h} &= \frac{\partial}{\partial x^\alpha} \left(\frac{\partial g}{\partial \Xi^A}(\Xi^J)\, \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) \right) \\
\frac{(\Xi^J - \Xi^{J-1})^A}{h} &= \frac{\partial}{\partial x^\alpha} \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F^{J-1}) v_i^J \right).
\end{aligned}
$$



Iterates (v, Ξ) constructed by solving the constrained variational problem
Given v^0, $\Xi^0 = (F^0, Z^0, w^0)$,

$$\min \int_{\mathbb{T}^3} \left(\frac{1}{2}|v - v^0|^2 + g(F, Z, w) \right) dx$$

over the affine subspace

$$\mathcal{C} := \Big\{ (v, F, Z, w) : \mathbb{T}^3 \to \mathbb{R}^{22} \text{ subject to the constraints}$$

$$\begin{aligned} \frac{1}{h}(F_{i\alpha} - F^0_{i\alpha}) &= \partial_\alpha v_i \,, \\ \frac{1}{h}(Z_{k\gamma} - Z^0_{k\gamma}) &= \partial_\alpha(\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F^0_{j\beta}v_i) \,, \\ \frac{1}{h}(w - w^0) &= \partial_\alpha((\mathrm{cof}\, F^0)_{i\alpha}v_i) \qquad \Big\}. \end{aligned}$$

Iterates decrease the mechanical energy, obey bounds

$$\sup_j \int_{\mathbb{T}} 3|v^j|^2 + g(\Xi^j)\, dx + \sum_j |v^j - v^{j-1}|^2_{L^2_x} + |\Xi^j - \Xi^{j-1}|^2_{L^2_x} \le E_0$$

Under coercivity for g and bounds for g and $\frac{\partial g}{\partial \Xi}$ we have

$$v^h \rightharpoonup v \quad \text{wk in } L^2$$
$$(F^h, Z^h, w^h) \rightharpoonup (F, Z, w) \quad \text{wk in } L^p \times L^q \times L^r$$

and (v, F) is a measure-valued solution of elasticity, which satisfies the weak from of the geometric transport identities Demoulini-Stuart-AT 01

Uniqueness of classical solutions for polyconvex elasticity within the class of measure-valued solutions Demoulini-Stuart-AT 11

The variational approximation scheme converges whenever the limiting solution is smooth. Miroshnikov-AT 11

Radial motions of isotropic elastic materials

Radial motions $y(x,t) = w(R,t)\frac{x}{R}$, $R = |x|$, $x \in \mathbb{R}^3$

$$w_{tt} = \frac{1}{R^2}\partial_R \left(R^2 \frac{\partial \Phi}{\partial v_1}(w_R, \frac{w}{R}, \frac{w}{R}) \right) - \frac{1}{R}\Big(\frac{\partial \Phi}{\partial v_2} + \frac{\partial \Phi}{\partial v_3} \Big)(w_R, \frac{w}{R}, \frac{w}{R})$$

- $W(F)$ frame indifferent, isotropic, elastic

$$W(F) = \Phi(v_1, v_2, v_2)$$

  with Φ symmetric function of eigenvalues v_1, v_2, v_3 of $\sqrt{F^T F}$ and polyconvex, *e.g.*

$$\Phi = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) + h(v_1 v_2 v_3) \quad \text{with } h(\delta) \to +\infty \text{ as } \delta \to 0+$$

- To represent a physically realizable motion: $\det F > 0$ with $F = \nabla y$.

$$\det F = w_R (w/R)^2 > 0$$

  Also sufficient condition for avoiding interpenetration of matter.

Existence and properties for energy minimizers, cavitating solutions

Ball '82, Sivaloganathan '86, ...

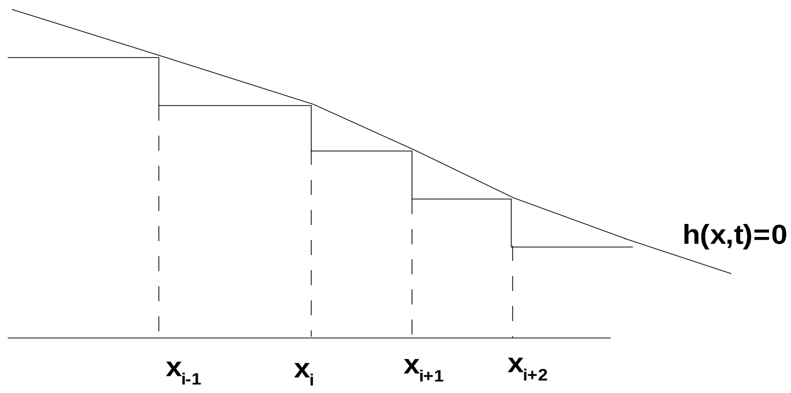
Dynamic, self-similar, cavitating solutions that decrease the total energy

Pericak-Spector '88, ...

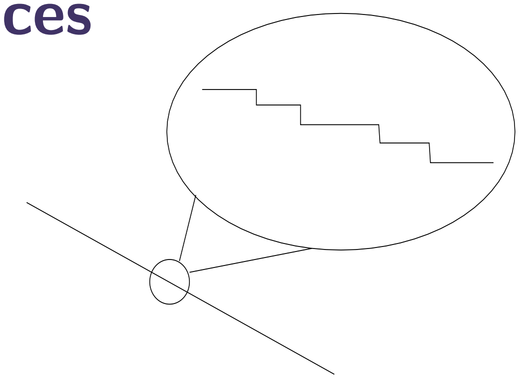
Construction of a variational approximation scheme, that preserves the positivity of Jacobians, and is stable and consistent with entropy dissipation.
(Scheme is based on null-Lagrangeans and extensions of the radial elastodynamics system)

Miroshnikov-AT '11

Mesoscopic theories for crystalline surfaces



(a) Monotonic surface consisting of steps



(b) magnifying lens view

Relaxation of surface to equilibrium below the roughening transition

- Crystalline surface in local equilibrium with its vapor phase. Relaxation effected by adsorption-desorption processes (evaporation-condensation)
- Crystal in vacuum. Relaxation effected through breaking and reconstruction of bonds, process called surface diffusion

Continuum Thermodynamics models - Spohn - 93

- Evaporation-Condensation

$$\left.\begin{aligned} \frac{\partial h}{\partial t} &= -\kappa(\nabla h)\frac{\delta F}{\delta h} \\ F[h] &= \int dx\, \sigma(\nabla h) \end{aligned}\right\} \qquad \frac{\partial h}{\partial t} = \kappa(\nabla h)\sum_{\alpha}\partial_{x_\alpha}\frac{\partial \sigma}{\partial p_\alpha}(\nabla h)$$

- Surface Diffusion

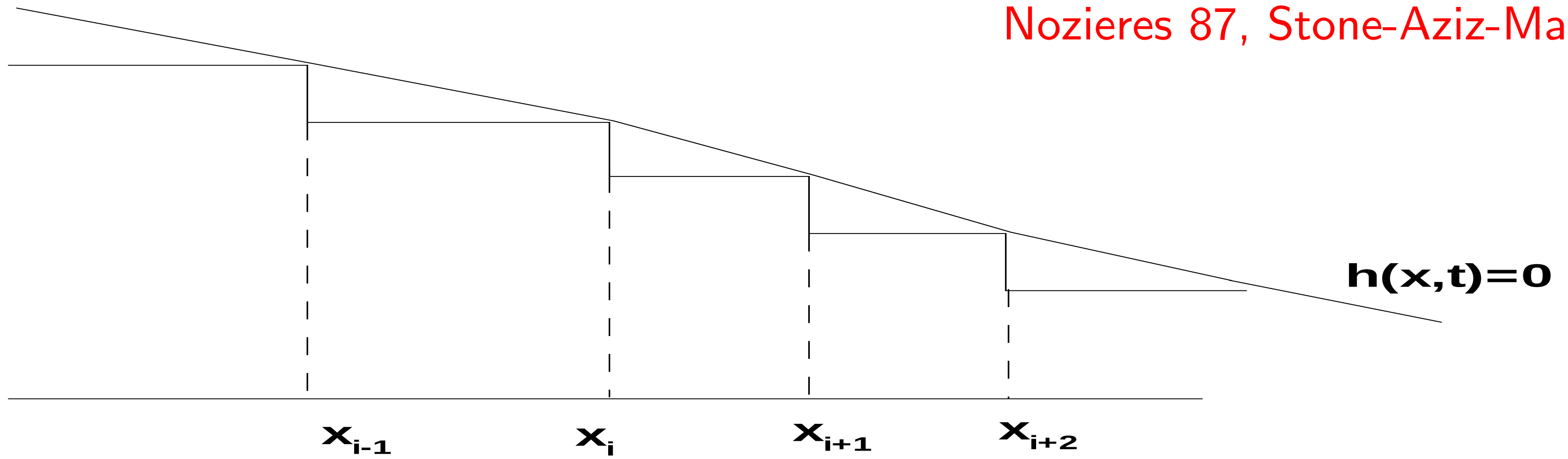
$$\left.\begin{aligned} \frac{\partial h}{\partial t} + \nabla\cdot j &= 0 \\ j &= -\kappa(\nabla h)\nabla\frac{\delta F}{\delta h} \\ F[h] &= \int dx\, \sigma(\nabla h) \end{aligned}\right\} \qquad \frac{\partial h}{\partial t} = -\nabla\cdot\kappa(\nabla h)\,\nabla\sum_{\alpha}\partial_{x_\alpha}\frac{\partial \sigma}{\partial p_\alpha}(\nabla h)$$

Mesoscopic step models - Burton-Cabrera-Franck theory - 50's for surface diffusion

mesoscopic theories vs continuum models

- Issue: Comparison between macroscopic theories and continuum thermodynamic models

Nozieres 87, Stone-Aziz-Margetis 05, ...



$$\dot{x}_i = -\frac{1}{\alpha}\left(g\Big(\frac{x_{i+1}-x_i}{\alpha}\Big) - g\Big(\frac{x_i - x_{i-1}}{\alpha}\Big)\right)$$

$$\dot{x}_i = -\frac{h_t}{h_x}$$

$$m_i = \frac{a}{x_{i+1}-x_i} \sim -h_x(x_i,t)$$

as $a \to 0$

$$-\frac{h_t}{h_x} = -\frac{1}{m}\partial_x\left[g\Big(\frac{1}{m}\Big)\right]$$

$$\partial_t m = \partial_{xx} g\Big(\frac{1}{m}\Big)$$

Kinetic Description of ODE System

Evaporation-Condensation

$$\dot{x}_i = -\frac{1}{\alpha}\left[g\big(\frac{x_{i+1}-x_i}{\alpha}\big) - g\big(\frac{x_i - x_{i-1}}{\alpha}\big)\right]$$

$$x_1 < ... < x_{i-1} < x_i < x_{i+1} < ... < x_N \qquad aN = 1$$

Objectives: 1/. Develop a kinetic description

$$F_1^a(y,t) = a\sum_{i=1}^{N} \delta(y - x_i(t))$$

$$F_2^a(y_1,y_2,t) = a\sum_{i=1}^{N} \delta(y_1 - x_i(t))\delta(y_2 - x_{i+1}(t))$$

$$F_n^a(y_1,...,y_n,t) = a\sum_{i=1}^{N} \delta(y_1 - x_i(t))\delta(y_2 - x_{i+1}(t))...\delta(y_n - x_{i+n-1}(t))$$

BBGKY Hierarchy

$$F_1^a(x,t) = a\sum_{i=1}^{N}\delta(x - x_i(t))$$

$$F_n^a(y_1, ..., y_n, t) = a\sum_{i=1}^{N}\delta(y_1 - x_i(t))\delta(y_2 - x_{i+1}(t))...\delta(y_n - x_{i+n-1}(t))$$

The hierarchy for F_n^a is described by

$$n = 1 : \quad \partial_t F_1^a(x) + \partial_x\left\{\int \mathrm{d}z\,\mathrm{d}y\;\frac{1}{a}\left[g\left(\frac{x-z}{a}\right) - g\left(\frac{y-x}{a}\right)\right] F_3^a(z,x,y)\right\} = 0\ , \tag{1}$$

$$\begin{aligned} n = 2 : \quad &\partial_t F_2^a(x,y) + \partial_x\left\{\int \mathrm{d}z\;\frac{1}{a}\left[g\left(\frac{x-z}{a}\right) - g\left(\frac{y-x}{a}\right)\right] F_3^a(z,x,y)\right\} \\ &+ \partial_y\left\{\int \mathrm{d}z\;\frac{1}{a}\left[g\left(\frac{y-x}{a}\right) - g\left(\frac{z-y}{a}\right)\right] F_3^a(x,y,z)\right\} = 0\ , \end{aligned} \tag{2}$$

$$\begin{aligned} n \geq 3 : \quad &\partial_t F_n^a(\vec{y}) + \partial_{y_1}\left\{\int \mathrm{d}p\;\frac{1}{a}\left[g\left(\frac{y_1-p}{a}\right) - g\left(\frac{y_2-y_1}{a}\right)\right] F_{n+1}^a(p,\vec{y})\right\} \\ &+ \sum_{j=2}^{n-1}\partial_{y_j}\left\{\frac{1}{a}\left[g\left(\frac{y_j-y_{j-1}}{a}\right) - g\left(\frac{y_{j+1}-y_j}{a}\right)\right] F_n^a(\vec{y})\right\} \\ &+ \partial_{y_n}\left\{\int \mathrm{d}p\;\frac{1}{a}\left[g\left(\frac{y_n-y_{n-1}}{a}\right) - g\left(\frac{p-y_n}{a}\right)\right] F_{n+1}^a(\vec{y},p)\right\} = 0\ . \end{aligned} \tag{3}$$

Macroscopic Limit

2/ Macroscopic description: $a \to 0$, $aN = 1$

Thm Suppose that

$$\zeta_i(x_{i-2}, x_{i-1}, x_i, x_{i+1}, x_{i+2}; a) = v(x_i) + o(1)$$

uniformly on i, as $a \to 0$, $N \to \infty$, $Na = 1$.

Then $F_n^a \rightharpoonup F_n$ and F_n satisfies

$$\partial_t F_1 + \partial_y (v(y) F_1) = 0$$

$$\partial_t F_n(y_1, ..., y_n, t) + \sum_{j=1}^{n} \partial_{y_j} \Big(v(y_j) F_n(y_1, ..., y_n, t) \Big) = 0$$

Margetis-AT '09

example: Evaporation-Condensation. $v = \frac{1}{F_1} \partial_y \left(g \circ \frac{1}{F_1} \right)$

$$\partial_t F_1 + \partial_y \left(\frac{\partial}{\partial y} (g \circ \frac{1}{F_1}) \right) = 0$$

3/ Correlation functions

$$\partial_t F_1(y,t) + \partial_y \big(v(y,t)F_1\big) = 0$$
$$\partial_t F_2(x,y,t) + \partial_y \big(v(x,t)F_2\big) + \partial_z \big(v(y,t)F_2\big) = 0$$

High order correlation functions are expressed in terms of F_1 via the formulas

$$F_2(x,y,t) = F_1(x,t)F_1(y,t)C(h(x,t),h(y,t))$$
$$F_n(y_1,...y_n,t) = F_1(y_1,t)...F_1(y_n,t)\ C(h(y_1,t),...,h(y_n,t))$$

where $h_x = -F_1$ and the shape C captures the correlation of the initial data.

Formula describes the evolution of correlation functions
contrast to propagation of chaos