Multi-dimensional Degenerate Keller-Segel system with new diffusion exponent $2n/(n+2)$

CHEN, Li

Department of Mathematical Sciences, Tsinghua University, Beijing, China.
http://faculty.math.tsinghua.edu.cn/faculty/~lchen/

November 28, 2011

This is a joint work with Jian-Guo Liu and Jinhuan Wang

1. Keller-Segel system in chemotaxis
   - 2-D result and critical mass
   - Some generalization in Multi-D
2. Stationary solutions
   - Stationary solutions in 2-D
   - Stationary solutions in Multi-D
3. Degenerate system with new exponent $2n/(n+2)$
   - Variational structure and entropy inequality
   - Conservation relation for moments
   - Steady state and Hardy-Littlewood-Sobolev inequality
4. Radially symmetric solutions
   - Subcritical case and long time decay
   - Supercritical case and blow up
5. Existence and blow up for general initial data
   - Existence for some initial data
   - Blow up for supercritical case

2-D result and critical mass

The Keller-Segel system was widely studied in the literature since 1970's.
We will focus on the simplified version,

$$\begin{aligned} \rho_t &= \Delta\rho - \mathrm{div}(\rho\nabla c), && x \in \mathbb{R}^2, t \geq 0, \\ -\Delta c &= \rho, && x \in \mathbb{R}^2, t \geq 0, \\ \rho(x,0) &= \rho_0(x), && x \in \mathbb{R}^2. \end{aligned}$$

where $\rho(x,t)$ represents the bacteria density, $c(x,t)$ represents the chemical substance concentration.

Typical quantities of the system

- Conservation of mass

$$m_0(t) = \int_{\mathbb{R}^2} \rho(x,t)dx = \int_{\mathbb{R}^2} \rho_0(x)dx.$$

- Entropy inequality

$$\frac{d}{dt}F(\rho) + \int_{\mathbb{R}^2} \rho|\nabla \ln \rho - \nabla c|^2 dx \leq 0.$$

where $F(\rho) = \int_{\mathbb{R}^2}(\rho \ln \rho - \frac{\rho c}{2})dx$ and

$$\int_{\mathbb{R}^2} \frac{\rho c}{2} dx = \frac{1}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \rho(x,t)\rho(y,t) \frac{1}{\log|x-y|^2} dxdy.$$

Logarithmic Hardy-Littlewood-Sobolev inequality

Let f be a nonnegative function in $L^1(\mathbb{R}^2)$ such that $f\log f \in L^1(\mathbb{R}^2)$, then

$$\int_{\mathbb{R}^2} f(x)\log f(x)dx - \frac{2}{M}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)f(y)\log\frac{1}{|x-y|}dxdy + C(M) \geq 0.$$

where $M = \int_{\mathbb{R}^2} f(x)dx$, $C(M) := M(1+\log\pi - \log M)$.

Recall that the entropy is

$$F(\rho) = \int_{\mathbb{R}^2}\rho\ln\rho dx - \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x,t)\rho(y,t)\frac{1}{\log|x-y|^2}dxdy,$$

Then by Log H-L-S inequality, either

$$F(\rho(\cdot,t)) \geq (1-\frac{m^0}{8\pi})\int_{\mathbb{R}^2}\rho\log\rho dx - \frac{m^0}{8\pi}C(m^0),$$

or

$$F(\rho(\cdot,t)) \geq (\frac{1}{m^0}-\frac{1}{8\pi})\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\rho(x,t)\rho(y,t)\log\frac{1}{|x-y|^2}dxdy - C(m^0).$$

★ $m_0 < 8\pi$, global existence, A. Blanchet, J. Dolbeault and B. Perthame in 2006.
We (Carrillo, Chen, Liu and Wang) gave a new proof based Delort's theory on 2-D incompressible Euler equation, to appear in *Acta Applicanda Mathematicae.*

Critical mass and blow up discussion was given by J. Dolbeault and B. Perthame in 2004.

Second moment

$$m_2(t) := \int_{\mathbb{R}^2} \frac{|x|^2}{2} \rho(x,t)dx.$$

In 2-D, the time derivative of second moment is

$$\frac{d}{dt} m_2(t) = 2m_0(1 - \frac{m_0}{8\pi}).$$

★ $m_0 > 8\pi$, one can conclude that $m_2(t)$ should become negative in finite time which is impossible since ρ is nonnegative. Therefore the solution can't be smooth until that time.

Critical mass 8π in 2-D

- $m_0 < 8\pi$, global existence,
- $m_0 > 8\pi$, finite time blow up.
- $m_0 = 8\pi$,
    - Blanchet, Carrillo, Masmoudi, Global existence and infinite blow up of free energy solution, with constant second momentum.
    - Blanchet, Carlen and Carrillo, By making full use of the gradient flow structure and relative entropy, they give conditions for initial data to belong to the basin of attraction for each of the infinitely many stationary solutions.
    - ......

Some generalization in Multi-D

- Use $-\log|x|$ instead of $\dfrac{1}{|x|^{n-2}}$. Calvez, Perthame and Sharifi, 2007.
- Use degenerate diffusion or nonlinear advection to balance the nonlocal aggregation effect.

$$\rho_t = \Delta\rho^m - \mathrm{div}(\rho^{q-1}\nabla c), \qquad x \in \mathbb{R}^n,\ t \geq 0,$$

there is a so called critical exponent in the literature $m^* = q - \frac{2}{n}$. m^* is the exponent that if $u(x,t)$ is a solution, then

$$u_\lambda(x,t) = \lambda^n u(\lambda x, t)$$

is still a solution. It is also similar to the *Fujita* exponent in nonlinear parabolic equations.

Some known discussions,

- $m = 1$, a critical exponent $m = 1 = q - 2/n$. Horstmann and Winkler, 2005.
  - $q < 1 + 2/n$, global solution for large initial data.
  - $q > 1 + 2/n$, unbounded solution for **some** initial data.
- $q = 2$, a critical exponent $m^* = 2 - 2/n$. Sugiyama 2006.
  - $m > m^*$, global existence. diffusion dominates,
  - $m < m^*$, finite time blow up, **some** initial data,
- $q = 2$, $m = m^*$, Blanchet, Carrillo and Laurencot, 2009.
  - $m_0 < M_c$ and $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$, global weak solution exists and satisfies an energy-dissipation inequality.
  - $m_0 > M_c$, $\rho_0 \in L^\infty \cap H^1(\mathbb{R}^n)$ and $F(\rho_0) < 0$, finite time blow up for the solution in $L^m(\mathbb{R}^n)$.
  - $m_0 = M_c$, large time behavior with special initial data.

More results related to $m^* = q - 2/n$.

- $m < q - 2/n$, small initial data, long time behaves like Barenblatt solution of the porous media. Luckhaus and Sugiyama , 2006.
- $m > 3 - 4/n \geq m^* = 2 - 2/n$, global existence of nonnegative weak solution, Kowalczyk and Szymańska, 2008.
- $q = 2$, $m > 2 - 2/n$, global classic solution for any L^∞ initial data, Cieślak and Laurencot, 2009.
- $m > q - 2/n$, global existence of weak solution with large initial data, Ishida and Yokota, 2011.
- critical exponent for general interaction potentials, Bedrossian, Rodríguez and Bertozzi,2011.
- ......

QUESTION for $m = m^*$

No nontrivial nonnegative stationary solution!

Stationary solutions in 2-D

$$-\Delta c = e^c, \text{ in } \mathbb{R}^2, \tag{1}$$

(1) has a family of solutions

$$C_{\lambda,x^0}(x) = \log\left[8\left(\frac{\lambda}{\lambda^2+|x-x_0|^2}\right)^2\right], \quad \forall \lambda > 0, x^0 \in \mathbb{R}^2.$$

Back to the equation, the stationary solution for ρ is

$$U_{\lambda,x^0}(x) = e^{C_{\lambda,x^0}(x)},$$

and

$$\int_{\mathbb{R}^2} U_{\lambda,x^0}(x)dx = 8\pi.$$

Stationary solutions in Multi-D

$$-\Delta c = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} c^{\frac{1}{m-1}}, \text{ in } \mathbb{R}^n, \tag{2}$$

By Gidas, Spruck's result in 1981, when $1 \le \frac{1}{m-1} < \frac{n+2}{n-2}$, the only nonnegative solution is 0.
In the case $\frac{1}{m-1} = \frac{n+2}{n-2}$, (2) has a family of solutions

$$C_{\lambda,x_0}(x) = \frac{2^{\frac{n+2}{4}} n^{\frac{n}{2}}}{n-2} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}, \quad \forall \lambda > 0, x_0 \in \mathbb{R}^n.$$

W. Chen and C. Li prove the result by moving plane method in 1991.

The stationary solution

$$U_{\lambda,x_0}(x) = \left(\frac{m-1}{m}\right)^{\frac{1}{m-1}} C_{\lambda,x_0}^{\frac{1}{m-1}}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2}\right)^{\frac{n+2}{2}}.$$

with L^m norm independent of λ and x_0,

$$\|U_{\lambda,x_0}\|_{L^m}^m = \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n}{2}}$$

where

$$C(n) = \pi^{(n-2)/2} \frac{1}{\Gamma(n/2+1)} \left\{\frac{\Gamma(n/2)}{\Gamma(n)}\right\}^{-2/n}.$$

Note: $\frac{1}{m^*-1} < \frac{n+2}{n-2}$, so the only nonnegative nontrivial solution is 0.

Stationary solutions in 2-D and M-D have the uniform formula

$$U_{\lambda,x_0}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x-x_0|^2} \right)^{\frac{n+2}{2}}.$$

and uniform formula for L^m norm, $m=1$ in the case of $n=2$

$$\|U_{\lambda,x_0}\|_{L^m}^m = \left(\frac{2n^2\alpha(n)}{C(n)} \right)^{\frac{n}{2}}, \text{ which is } 8\pi \text{ when } n=2.$$

Moreover, the second moment of stationary solution is infinity

$$\int_{\mathbb{R}^n} |x-x_0|^2 U_{\lambda,x_0}(x) = \infty.$$

New exponent $2n/(n+2)$

Now we choose

$$\frac{1}{m-1} = \frac{n+2}{n-2}$$

which is exactly

$$m = \frac{2n}{n+2}.$$

Degenerate system with new exponent

We will study the system when $m = \frac{2n}{n+2}$,

$$\begin{aligned} \rho_t = \Delta \rho^m - \mathrm{div}(\rho \nabla c), &\quad x \in \mathbb{R}^n, \ t \geq 0, \\ -\Delta c = \rho, &\quad x \in \mathbb{R}^n, \ t \geq 0, \\ \rho(x,0) = \rho_0(x), &\quad x \in \mathbb{R}^n \end{aligned}$$

where c can be represented by fundamental solution,

$$c(x,t) = \frac{1}{(n-2)n\alpha(n)} \int_{\mathbb{R}^n} \frac{\rho(y,t)}{|x-y|^{n-2}} dy,$$

Free energy

$$\mathcal{F}(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{1}{2}\int_{\mathbb{R}^n} \rho(x,t)c(x,t)dx.$$

or

$$\mathcal{F}(\rho) = \frac{1}{m-1}\int_{\mathbb{R}^n} \rho^m(x,t)dx - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy.$$

where $c_n = \dfrac{1}{(n-2)n\alpha(n)}$.

Key feature of the system

The different sign in above free energy represents the competition between diffusion and nonlocal aggregation.

Variational structure

The first order variation of $\mathcal{F}$ gives the chemical potential:

$$\mu = \frac{\delta \mathcal{F}}{\delta \rho} = \frac{m}{m-1}\rho^{m-1} - c.$$

By defining the drift velocity $v = -\nabla\mu$, the equation can be rewritten into a continuity equation:

$$\rho_t + \mathrm{div}(\rho v) = 0,$$

or

$$\rho_t = \mathrm{div}\left(\rho\nabla\left(\frac{m}{m-1}\rho^{m-1} - c\right)\right).$$

Energy dissipation relation

Take inner product of $\frac{\delta \mathcal{F}}{\delta u}$ the equation, one leads to the following

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho |\nabla \mu|^2 dx = 0,$$

or

$$\frac{d\mathcal{F}(\rho)}{dt} + \int_{\mathbb{R}^n} \rho \left| \nabla \left(\frac{m}{m-1} \rho^{m-1} - c \right) \right|^2 dx = 0,$$

which leads to the fact that $\mathcal{F}(\rho(\cdot, t))$ is a monotone nonincreasing function of t.

Conservation relation for moments

The ith-moment of ρ, $i = 0, 1, 2$, is defined by

$$m_0(t) = \int_{\mathbb{R}^n} \rho(x,t)dx, m_1(t) = \int_{\mathbb{R}^n} x\rho(x,t)dx, m_2(t) = \int_{\mathbb{R}^n} |x|^2\rho(x,t)dx.$$

Direct computation implies,

$$\begin{aligned} m_0'(t) &= \frac{d}{dt}\int_{\mathbb{R}^n} \rho(x,t)dx = 0, \\ m_1'(t) &= \frac{d}{dt}\int_{\mathbb{R}^n} x\rho(x,t)dx = 0, \\ m_2'(t) &= -4\int_{\mathbb{R}^n} \rho^m(x,t)dx + 2(n-2)\mathcal{F}(\rho(\cdot,t)). \end{aligned}$$

Steady state revisited, for some $\lambda > 0, x_0 \in \mathbb{R}^n$,

$$C_{\lambda,x_0}(x) = \frac{2^{\frac{n+2}{4}} n^{\frac{n}{2}}}{n-2} \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}.$$

$$U_{\lambda,x_0}(x) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n+2}{2}}.$$

L^m norm of stationary solution

$$\|U_{\lambda,x_0}(x)\|_{L^m}^m \text{ is a constant independent of } \lambda, x_0.$$

If $\lim_{t\to\infty} \rho(x,t) = U_{\lambda,x_0}(x)$, then the parameters $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are uniquely determined by m^0 and m^1,

$$x_0 = m_1/m_0, \quad \lambda^{\frac{n-2}{2}} \frac{2\pi}{n} \left(\frac{n-2}{2n}\right)^{\frac{n+2}{n-2}} [n(n-2)]^{\frac{n+2}{4}} = m_0.$$

A special version of Hardy-Littlewood-Sobolev inequality is given for $\rho \in L^m(\mathbb{R}^n)$, it holds

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\rho(x)\rho(y)}{|x-y|^{n-2}} dxdy \leq C(n)\|\rho\|_{L^m}^2,$$

where

$$C(n) = \pi^{(n-2)/2} \frac{1}{\Gamma(n/2+1)} \left\{ \frac{\Gamma(n/2)}{\Gamma(n)} \right\}^{-2/n}. \tag{3}$$

Moreover, the equality holds if and only if $\rho(x) = AU_{\lambda,x_0}(x)$, for some constant A and parameters $\lambda > 0$, $x_0 \in \mathbb{R}^n$.

Decomposition of free energy

$$
\begin{aligned}
\mathcal{F}(\rho) &= \frac{1}{m-1}\int_{\mathbb{R}^n}\rho^m(x,t)dx - \frac{c_n}{2}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{\rho(x,t)\rho(y,t)}{|x-y|^{n-2}}dxdy,\\
&= \frac{1}{m-1}\|\rho\|_{L^m}^m\left(1-\frac{(m-1)c_nC(n)}{2}\|\rho\|_{L^m}^{4/(n+2)}\right)\\
&\quad +\frac{c_n}{2}\left(C(n)\|\rho\|_{L^m}^2-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right)\\
&:= \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho).
\end{aligned}
$$

Since $U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$, it is also a critical point for $\mathcal{F}_1(\rho)$. Indeed we will show that it is a maximum point for $\mathcal{F}_1(\rho)$. This property will be used in the proof of a finite time blow up.

Conformal invariants of free energy

By direct calculation, we can obtain the free energy $\mathcal{F}(\rho)$ is invariant under conformal mapping.

1. $\mathcal{F}(\rho_{\bar{x}}) = \mathcal{F}(\rho)$ with $\rho_{\bar{x}}(x) := \rho(x+\bar{x})$, $\forall \bar{x} \in \mathbb{R}^n$;
2. $\mathcal{F}(\rho_\lambda) = \mathcal{F}(\rho)$ with $\rho_\lambda(x) := \lambda^{\frac{n+2}{2}} \rho(\lambda x)$, $\forall \lambda > 0$;
3. $\mathcal{F}(\rho_{\mathcal{R}}) = \mathcal{F}(\rho)$ with $\rho_{\mathcal{R}}(x) := \rho(\mathcal{R}^{-1}x)$, $\forall\ \mathcal{R}^*\mathcal{R} = I$;
4. $\mathcal{F}(\rho_{\bar{x},\lambda}) = \mathcal{F}(\rho)$ with
$$\rho_{\bar{x},\lambda}(x) := \left(\frac{\lambda}{|x-\bar{x}|}\right)^{n+2} \rho\left(\bar{x} + \frac{\lambda^2(x-\bar{x})}{|x-\bar{x}|^2}\right),\ \forall \bar{x} \in \mathbb{R}^n,\ \lambda > 0.$$

Liouville's theorem implies that any smooth conformal mapping on a domain of $\mathbb{R}^n$, $n > 2$, can be expressed as a composition of translations, similarities, orthogonal transformations and Kelvin transformations (or inversions).

These transformations are all Möbius transformations.

Radially symmetric solutions

Theorem

Assume that the initial data $\rho_0 \geq 0$ is radially symmetric,

1. *If* $\exists \lambda_0 > 0$ *s.t.*
$$\rho_0(r) < U_{\lambda_0}(r), \quad \forall r \in [0, +\infty),$$
then any radially symmetric solution $\rho(r,t)$ is vanishing in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$.

2. *If* $\exists \lambda_0 > 0$ *s.t.*
$$\rho_0(r) > U_{\lambda_0}(r), \quad \forall r \in [0, +\infty),$$
then any radially symmetric solution $\rho(r,t)$ must blow up at a finite time t^* or has a mass concentration at $r = 0$ as time goes to infinity in the sense that there is $r(t) \to 0$ as $t \to \infty$ and a positive constant C such that
$$\int_{B(0,r(t))} \rho dx \geq C.$$

Idea We work on the following variable

$$M(t,r) := n\alpha(n) \int_0^r \sigma^{n-1} \rho(t,\sigma) d\sigma$$

by the $-\Delta c = \rho$, one has $M(t,r) = -n\alpha(n) r^{n-1} c'$.
Then the whole system can be reduced to a single equation for $M(t,r)$.

$$\begin{cases} M_t = n\alpha(n) r^{n-1} \left[\left(\dfrac{M'}{n\alpha(n) r^{n-1}} \right)^m \right]' + \dfrac{M' M}{n\alpha(n) r^{n-1}}, \\ M(t,0) = 0, M(t,\infty) = m_0, \\ M(0,r) = n\alpha(n) \displaystyle\int_0^r \sigma^{n-1} \rho_0(\sigma) d\sigma. \end{cases}$$

Stationary radially symmetric solution

$$U_\lambda(r) = 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \Big(\frac{\lambda}{\lambda^2 + r^2}\Big)^{\frac{n+2}{2}},$$
$$C_\lambda(r) = 2^{\frac{n+2}{4}} n^{\frac{n}{2}} (n-2)^{-1} \Big(\frac{\lambda}{\lambda^2 + r^2}\Big)^{\frac{n-2}{2}}$$

and

$$\tilde{M}_\lambda(r) \quad = n\alpha(n) \int_0^r \sigma^{n-1} U_\lambda(\sigma) d\sigma = K_\lambda(n) \frac{1}{(1+\lambda^2 r^{-2})^{\frac{n}{2}}},$$

where $K_\lambda(n) = \alpha(n) 2^{\frac{n+2}{4}} n^{\frac{n+2}{2}} \lambda^{\frac{n-2}{2}}$.

Subcritical case and long time decay

Lemma

For $n \geq 3$, assume that

$$m_0 = M(t,\infty) < K_{\lambda_0}(n),\ \ M(0,r) < \tilde{M}_{\lambda_0}(r), \forall r > 0.$$

for some $\lambda_0 > 0$. Then the solutions diminish in time in the following sense

$$M(t,r) \to 0 \text{ as } t \to \infty \text{ uniformly on any interval } 0 \leq r \leq R,$$

and thus $\rho(t,x)$ vanishes in $L^1_{loc}(\mathbb{R}^n)$ as $t \to \infty$.

Idea: Construction of super solution

$\exists \mu \in (0,1)$ s.t. $M(0,r) \leq \mu \tilde{M}_{\lambda_0}(r)$.
The super-solution $\bar{N}(t,r)$ is given by

$$\bar{N}(t,r) = \min\left\{ m_0, \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}} \right\} = \begin{cases} m_0 & r > R(t) \\ \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)r^{-2})^{n/2}}, & r \leq R(t) \end{cases}$$

Motivation for super-solution: Use stationary solution

$$\tilde{M}_\lambda(r) = K_\lambda(n)\frac{1}{(1+\lambda^2 r^{-2})^{\frac{n}{2}}},$$

and modifying constant $\lambda = \lambda(t) = (A_1 t + \lambda_0^n)^{1/n}$, for some $A_1 > 0$. Then cut it off by a constant m_0 for $r \geq R(t)$. $R(t)$ is determined by

$$m_0 = \frac{\mu K_{\lambda_0}(n)}{\left(1+\lambda^2(t)R^{-2}(t)\right)^{n/2}}.$$

Direct and calculations show that, by choosing

$$A_1 = A_0(\mu K_{\lambda_0}(n))^{m-2} m_0 \alpha(n)^{-1} \left[\left(\frac{\mu K_{\lambda_0}(n)}{m_0} \right)^{\frac{2}{n}} - 1 \right]^{n/2} > 0,$$

where $A_0 = 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1-\mu^{2-m})$, $\bar{N}(t,r)$ is a super solution. By the comparison principle, we deduce that the solution $M(t,r) \le \bar{N}(t,r)$ in $[0,\infty)\times[0,\infty)$. Notice that $\lambda(t), R(t) \to \infty$ as $t \to \infty$. So, for a given interval $r \in (0, R_0)$, it holds that

$$M(t,r) \le \frac{\mu K_{\lambda_0}(n)}{(1+\lambda^2(t)R_0^{-2})^{n/2}} \to 0, \text{ as } t \to \infty.$$

Supercritical case and blow up

Lemma

For dimension $n \geq 3$. Assume that

$$m_0 = M(t,\infty) > K_{\lambda_0}(n), \quad M(0,r) > \tilde{M}_{\lambda_0}(r), r > 0,$$

for some $\lambda_0 > 0$ and there is no finite time blow up. Then there is $r(t) \to 0$ as $t \to \infty$ and $C > 0$ such that all solutions $M(r,t)$ satisfy

$$M(r(t),t) \geq C.$$

Or equivalently radially symmetric solutions ρ have mass concentration at $x = 0$, i.e.

$$\int_{B(0,r(t))} \rho dx \geq C.$$

We will show that there exists a radius $r(t) > 0$ depends on t such that as $t \to \infty$, we have $r(t) \to 0$ and

$$M(t, r(t)) \geq \text{ Const.} > 0, \tag{4}$$

i.e.,

$$\int_{B(0,r(t))} \rho dx \geq C > 0.$$

Idea of the proof.

we can choose $\mu_0 > 1$ such that $\mu_0 K_{\lambda_0}(n) < m_0 = M(t, \infty)$ and $M(0, r) > \mu_0 \tilde{M}_{\lambda_0}(r)$ where $\tilde{M}_{\lambda_0}(r)$ is the stationary solution. Construct a sub-solution,

$$\underline{N}(t, r) = \max\left\{\frac{m_0}{(1+\lambda_0^2 r^{-2})^{n/2}}, \frac{\mu_0 K_{\lambda_0}(n)}{(1+\lambda(t)^2 r^{-2})^{n/2}}\right\},$$

where $\lambda(t) = \lambda_0 e^{B_1 t}$ for some $B_1 < 0$.

Direct and calculations show that, by choosing

$$B_1 = B_0(\mu_0 K_{\lambda_0}(n))^m (m_0 n\alpha(n))^{-1} R_0^{-n} < 0.$$

where $B_0 = 2n^2\alpha(n)^{2-m}\lambda_0^{\frac{2(n-2)}{n+2}}(1-\mu_0^{2-m}) < 0$ and R_0 is determined by $\frac{m_0}{(1+\lambda_0^2 R_0^{-2})^{n/2}} = \mu_0 K_{\lambda_0}(n)$, $\underline{N}(t,r)$ is a sub-solution.
Now $\forall t > 0$, we have

$$M(t,r) \geq \underline{N}(t,r) \geq \frac{K_{\lambda_0}(n)}{[1+(\lambda_0 e^{B_1 t})^2 r^{-2}]^{n/2}}.$$

Furthermore we can take $r(t) = \lambda_0 e^{B_1 t} \to 0$ as $t \to +\infty$, then

$$M(t,r(t)) \geq \underline{N}(t,r(t)) = \frac{K_{\lambda_0}(n)}{2^{n/2}} > 0.$$

Existence and blow up for general initial data

Existence for some initial data

$$\begin{aligned} \rho_t &= \Delta\rho^m - \mathrm{div}(\rho\nabla c), && x \in \mathbb{R}^n,\ t \geq 0, \\ -\Delta c &= \rho, && x \in \mathbb{R}^n,\ t \geq 0, \\ \rho(x,0) &= \rho_0(x), && x \in \mathbb{R}^n \end{aligned}$$

Denote

$$C_s = \Big(\frac{4m^2}{(2m-1)^2 C_{GNS}}\Big)^{\frac{1}{2-m}} < \|U_{\lambda,x_0}\|_{L^m}.$$

Theorem

For initial date $\rho_0 \in L^1_+ \cap L^m$ and $\|\rho_0\|_{L^m} < C_s$, there is a global weak solution. Moreover $\|\rho(\cdot,t)\|_{L^m}$ decays algebraically,

$$\|\rho(\cdot,t)\|_{L^m} \leq Ct^{-\frac{1}{m(\beta-1)}}, \qquad \text{for large } t,$$

where $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Regularized problem

For small $\varepsilon > 0$,

$$\begin{aligned} \partial_t \rho_\varepsilon &= \Delta \rho_\varepsilon^m + \varepsilon \Delta \rho_\varepsilon - \mathrm{div}(\rho_\varepsilon \nabla c_\varepsilon), && x \in \mathbb{R}^n, t \geq 0, \\ -\Delta c_\varepsilon &= J_\varepsilon * \rho_\varepsilon && x \in \mathbb{R}^n, t \geq 0, \\ \rho(x,0) &= \rho_0(x), && x \in \mathbb{R}^n. \end{aligned}$$

where J_ε is a mollifier with radius ε. We know from parabolic theory that the above regularized problem has a global smooth positive solution u_ε for $t > 0$ if the initial data is nonnegative.

A priori estimates Taking $m\rho^{m-1}$ as a test function,

$$\frac{d}{dt}\int \rho^m dx + \frac{4m^2(m-1)}{(2m-1)^2}\int \left|\nabla \rho^{m-\frac{1}{2}}\right|^2 dx + \varepsilon \frac{4(m-1)}{m}\int \left|\nabla \rho^{\frac{m}{2}}\right|^2 dx$$

$$= \quad -(m-1)\int \nabla \rho^m \nabla c dx = (m-1)\int \rho^{m+1} dx.$$

The last term can be estimated by Gagliardo-Nirenberg-Sobolev inequality

$$(m-1)\left\|\rho^{m-\frac{1}{2}}\right\|_{L^{\frac{m+1}{m-\frac{1}{2}}}}^{\frac{m+1}{m-\frac{1}{2}}} \quad \leq \quad (m-1)C_{GNS}\left\|\nabla \rho^{m-\frac{1}{2}}\right\|_{L^2}^2 \|\rho\|_{L^m}^{2-m}$$

If we choose ρ_0 such that

$$(m-1)\left(-C_{GNS}\|\rho_0\|_{L^m}^{2-m} + \frac{4m^2}{(2m-1)^2}\right) := \delta > 0,$$

then we can obtain the estimate,

$$\frac{d}{dt}\int \rho^m dx + \delta \int \left|\nabla \rho^{m-\frac{1}{2}}\right|^2 dx \leq 0,$$

Strong convergence

From above estimates, we have $\rho \in L^\infty L^m \cap L^{m+1} L^{m+1}$, $\nabla \rho^{m-\frac{1}{2}} \in L^2 L^2$, On the other hand, Young's inequality implies

$$\|\nabla c\|_{L^\infty L^2} \leq C \|\rho\|_{L^\infty L^m} \leq C.$$

Thus by the equation itself, we have estimate for time derivative of ρ,

$$\partial_t \rho \in L_T^2 W^{-1,p}(U), \quad p\frac{2(m+1)}{m+3} > 1, \quad \text{for any bounded } U.$$

Moreover, by estimates in the cases of $3 \leq n < 6$ and $n \geq 6$,

$$\nabla \rho \in L_T^r L^r, \quad r = \min\{2, \frac{2(m+1)}{4-m}\}.$$

Then, Aubin's lemma implies strong convergence.

Long time algebraic decayBack to the estimates obtained before,

$$\frac{d}{dt}\int \rho^m dx \leq -\delta \int |\nabla \rho^{m-\frac{1}{2}}|^2 dx \leq -\frac{\delta}{C_{GNS}\|\rho\|_{L^m}^{2-m}} \int \rho^{m+1} dx.$$

On the other hand, we have

$$\|\rho\|_{L^m} \leq \|\rho\|_{L^{m+1}}^{\theta} \|\rho\|_{L^1}^{1-\theta}, \qquad \theta = \frac{m^2-1}{m^2},$$

Combining with the previous inequality, we have an inequality for $\|\rho\|_{L^m}$,

$$\frac{d}{dt}\int \rho^m dx \leq -\frac{\delta}{C_{GNS}} \|\rho\|_{L^m}^{m-2} \|\rho\|_{L^m}^{\frac{m^2}{m-1}} \|\rho\|_{L^1}^{\frac{1}{1-m}} = -C_d \left(\int \rho^m dx\right)^{\beta},$$

where $C_d = \frac{\delta}{C_{GNS}} \|\rho\|_{L^1}^{\frac{1}{1-m}}$, $\beta = \frac{2m^2-3m+2}{m(m-1)} > 1$.

Blow up for supercritical case

$\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$.

Decomposition of the free energy

$$\begin{aligned}\mathcal{F}(\rho) &= \frac{1}{m-1}\|\rho\|_{L^m}^m\left(1-\frac{(m-1)c_nC(n)}{2}\|\rho\|_{L^m}^{4/(n+2)}\right)\\ &\quad+\frac{c_n}{2}\left(C(n)\|\rho\|_{L^m}^2-\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{\rho(x)\rho(y)}{|x-y|^{n-2}}dxdy\right)\\ &:= \mathcal{F}_1(\rho)+\mathcal{F}_2(\rho).\end{aligned}$$

where $c_n = 1/(n(n-2)\alpha(n))$. By Hardy-Littlewood-Sobolev's inequality, $\mathcal{F}_2(\rho) \geq 0$.

$U_{\lambda,x_0}(x)$ is a critical point for both $\mathcal{F}(\rho)$ and $\mathcal{F}_2(\rho)$. Hence it is also a critical point for $\mathcal{F}_1(\rho)$.

$$\mathcal{F}_1(\rho) = f(\|\rho\|_{L^m}^m), \qquad f(s) = \frac{1}{m-1}s - \frac{c_n}{2}C(n)s^{\frac{2}{m}}$$

$f(s)$ is a strictly concave function, its unique maximum point at

$$s^* = \|U_{\lambda,x_0}\|_{L^m}^m = \left(\frac{2n^2\alpha(n)}{C(n)}\right)^{\frac{n}{2}}.$$

Assume $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$, $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m}$ and ρ is solution, then there is $\mu > 1$ such that ρ satisfy

$$\|\rho\|_{L^m}^m > \mu\|U_{\lambda,x_0}\|_{L^m}^m.$$

Theorem

Assume $m_2(0) = \int_{\mathbb{R}^n} |x|^2 \rho_0(x) dx < \infty$, $\mathcal{F}(\rho_0) < \mathcal{F}(U_{\lambda,x_0})$ and $\|\rho_0\|_{L^m} > \|U_{\lambda,x_0}\|_{L^m(\mathbb{R}^n)}$, then the solutions develop singularities, i.e., blow up in a finite time.

Proof. By directly calculation of $m_2'(t)$, we have

$$\begin{aligned} \frac{d}{dt} m_2(t) &\le -4\mu \|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(\rho_0) \\ &\le -4\mu \|U_{\lambda,x_0}\|_{L^m}^m + 2(n-2)\mathcal{F}(U_{\lambda,x_0}) \\ &= -4(\mu - 1)\|U_{\lambda,x_0}\|_{L^m}^m < 0, \end{aligned}$$

where we have used $\mathcal{F}(U_{\lambda,x_0}) = \frac{2}{n-2}\|U_{\lambda,x_0}\|_{L^m}^m$ in the third equality. This means that there is a $T > 0$ such that $\lim_{t\to T} m_2(t) = 0$.

On the other hand, $\forall R > 0$, by using Hölder inequality, we have

$$\int_{\mathbb{R}^n} \rho(x)dx \leq \int_{B_R} \rho(x)dx + \int_{B_R^c} \rho(x)dx \leq CR^{\frac{n-2}{2}} \|\rho\|_{L^m} + \frac{1}{R^2} m_2(t).$$

Now by choosing $R = (\frac{m_2(t)}{C\|\rho\|_{L^m}})^{2/(n+2)}$, we have

$$\|\rho\|_{L^1} \leq C\|\rho\|_{L^m}^{\frac{4}{n+2}} m_2(t)^{\frac{n-2}{n+2}}.$$

So,

$$\lim_{t\to T} \|\rho\|_{L^m}^{\frac{4}{n+2}} \geq \lim_{t\to T} \frac{\|\rho\|_{L^1}}{\bar{C}(n) m_2(t)^{\frac{n-2}{n+2}}} = \infty.$$

Future problems

Denote $m^c = \dfrac{2n}{2+n}$

- How about initial entropy larger than that of stationary solutions.
- Blow up behavior of the solution.
- Critical initial data $\|\rho_0\|_{L^{m^c}} = \|U_\lambda\|_{L^{m^c}}$
- General potential $\dfrac{1}{|x|^\lambda}$, $0 < \lambda \le n-2$, in progress.
- Diffusion exponent $m^c < m < m^*$.
- Nonlinear advection term $u^q \nabla c$...
- ......

THANK YOU!