

# An output-sensitive algorithm for computing projections of resultant polytopes

Christos Konaxis

Joint work with I.Z. Emiris, V. Fisikopoulos, and L. Peñaranda

Archimedes Center for Modeling, Analysis and Computation,  
University of Crete



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# Outline of the talk

- 1 Introduction
- 2 Resultant polytopes
- 3 The algorithm

# The problem

Given polynomials in  $K[x, y]$ :

$$f_0 = u_2x + u_1y + u_0$$

$$f_1 = c_{13}xy + c_{12}x + c_{11}y + c_{10}$$

$$f_2 = c_{23}xy^2 + c_{22}xy + c_{21}x + c_{20}$$

what is the condition that their coefficients  $(c_{10}, \dots, u_2)$  must satisfy so that the system  $f_0 = f_1 = f_2 = 0$  has a solution in  $(\mathbb{C}^*)^2$ ?

$$-c_{20}c_{21}u_1^3c_{12}c_{13}^2 - u_0c_{21}^2u_1^2c_{11}c_{13}^2 + c_{21}^2u_1^3c_{10}c_{13}^2 + \dots \text{another 57 terms} \dots = 0$$

When some of the polynomials' coefficients are specialized to constants,

$$f_0 = u_2x + u_1y + u_0$$

$$f_1 = 3xy + 11x + 4y + 1$$

$$f_2 = 2xy^2 + 5xy + x + 3$$

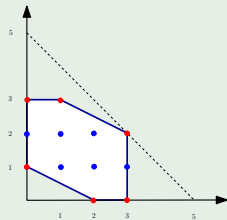
what is the condition that the remaining symbolic coefficients must satisfy so that the system  $f_0 = f_1 = f_2 = 0$  has a solution in  $(\mathbb{C}^*)^2$ ?

$$80u_0^3 - 136u_1u_0^2 - 180u_2u_0^2 + 288u_1^2u_0 - 674u_2u_1u_0 + 910u_2^2u_0 - 8u_1^3 - 62u_2u_1^2 + 183u_2^2u_1 - 250u_2^3 = 0$$

The **support**  $A(f)$  of a polynomial  $f$  is the set of exponents of its monomials with non-zero coefficient. The **Newton polytope**  $N(f)$  of  $f$  is the convex hull of its support.

## Example

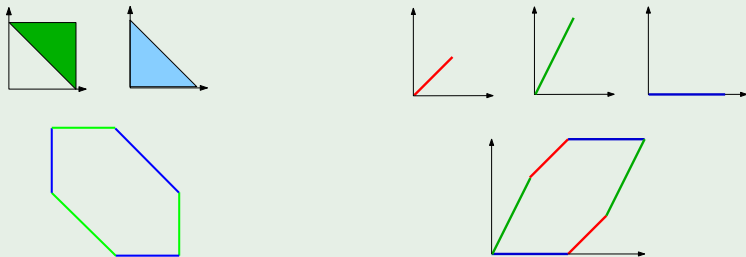
$$f(x, y) = 8y + xy - 24y^2 - 16x^2 + 220x^2y - 34xy^2 - 84x^3y + 6x^2y^2 - 8xy^3 + 8x^3y^2 + 8x^3 + 18y^3$$



The Newton polytope is the equivalent notion of total degree in sparse elimination theory.

The **Minkowski sum** of polytopes  $P_0, \dots, P_n$  (resp., supports  $A_0, \dots, A_n$ ) is the polytope  $P = P_0 + \dots + P_n := \{p_0 + \dots + p_n \mid p_i \in P_i\}$  (resp.,  $P = \text{conv}(A_0 + \dots + A_n) := \text{conv}\{a_0 + \dots + a_n \mid a_i \in A_i\}$ ).

## Example

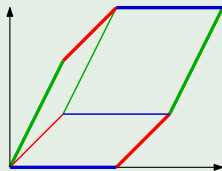
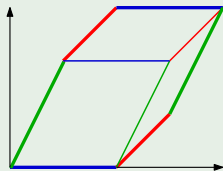


# Mixed subdivisions

A **fine mixed subdivision** of  $P = \text{conv}(A_0 + \cdots + A_n) \subset \mathbb{R}^n$ , is any family of cells partitioning  $P$  and intersecting properly as Minkowski sums of faces  $F_i \subset \text{conv}(A_i)$ , where  $\dim(\sum_i F_i) = \sum_i \dim F_i$ .

## Example

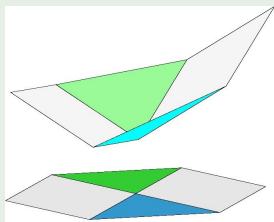
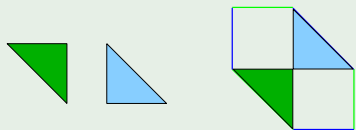
$$f_0 := c_{00} - c_{01}x_1x_2, \quad f_1 := c_{10} - c_{11}x_1x_2^2, \quad f_2 := c_{20} - c_{21}x_1^2$$



# Regular mixed subdivisions

A **regular** mixed subdivision of  $P = \text{conv}(A_0 + \cdots + A_n) \subset \mathbb{R}^n$  is obtained as the projection of the lower (or upper) hull of the Minkowski sum of lifted polytopes  $\hat{P}_i := \text{conv}\{(a_i, \omega_i(a_i)) \mid a_i \in A_i\} \subset \mathbb{R}^{n+1}$ . Generic  $\omega_i$  produce regular **fine** mixed subdivisions.

## Example





## Definition

The sparse resultant  $\mathcal{R}$  of polynomials  $f_0, \dots, f_n$ , where

$$f_i = \sum_{a \in A_i} c_{ia} x^a, \quad x^a = (x_1^{a_1}, \dots, x_n^{a_n}),$$

with symbolic  $c_{ia}$  and exponents in  $A_i \subset \mathbb{Z}^n$ , is the unique irreducible integer polynomial in the  $c_{ij}$ 's, which vanishes iff the  $f_i$  have a common complex root.

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## Example

Let  $A_0 = A_1 = \{0, 1\}$ . Then  $f_0 = c_{01}x + c_{02}$ ,  $f_1 = c_{11}x + c_{12}$ .

$$\mathcal{R} = c_{01}c_{12} - c_{02}c_{11} \in \mathbb{Z}[c_{01}, c_{02}, c_{11}, c_{12}]$$

# The sparse (toric) resultant

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$$\mathcal{R} = c_{01}c_{12} - c_{02}c_{11} \in \mathbb{Z}[c_{01}, c_{02}, c_{11}, c_{12}]$$

The Newton polytope  $N(\mathcal{R})$  of the resultant is called the **resultant polytope**.

## Theorem (Sturmfels'94)

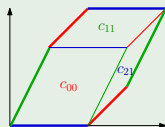
There exists a surjection  $\rho$  from the *regular fine mixed subdivisions* of  $P = \text{conv}(A_0 + \cdots + A_n)$  onto the *vertices* of  $N(\mathcal{R})$ . Subdivisions  $S, S'$  are *equivalent* iff  $\rho_S = \rho_{S'}$ .

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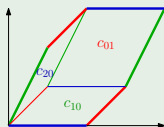
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## Example

$$f_0 := c_{00} - c_{01}st, \quad f_1 := c_{10} - c_{11}st^2, \quad f_2 := c_{20} - c_{21}s^2$$



$$c_{00}^4 c_{11}^2 c_{21}$$



$$c_{01}^4 c_{10}^2 c_{20}$$

$$\rho_S = (4, 0, 0, 2, 0, 1)$$

$$\rho_{S'} = (0, 4, 2, 0, 1, 0)$$

## Theorem

*Regular mixed subdivisions of  $P = \text{conv}(A_0 + \dots + A_n) \subset \mathbb{R}^n$  correspond bijectively to regular triangulations of  $\mathcal{A} := \bigcup_{i=0}^n (A_i \times \{e_i\}) \subset \mathbb{Z}^{2n}$ :  $e_i$  form an affine basis of  $\mathbb{R}^n$ .*

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$$f_0 = c_{01}x^2 + c_{02}y^2 + c_{03}x^2y^2$$

$$f_1 = c_{11}x^2 + c_{12}y^2 + c_{14}$$

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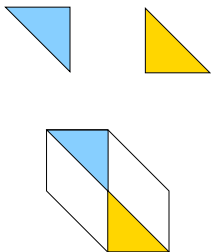


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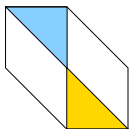
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$$f_0 = c_{01}x^2 + c_{02}y^2 + c_{03}x^2y^2$$

$$\mathcal{A} := \{(2, 0, 0), (0, 2, 0), (2, 2, 0)\},$$

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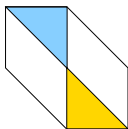
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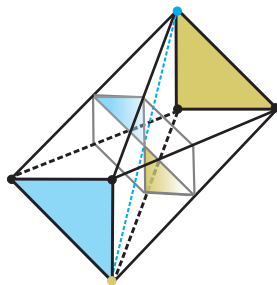
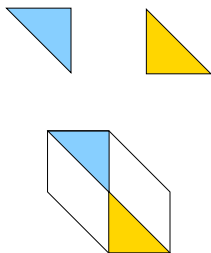
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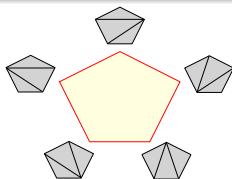
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## Theorem (Gelfand, Kapranov, Zelevinsky)

For every point set  $\mathcal{A} = \{a_1, \dots, a_{|\mathcal{A}|}\} \subset \mathbb{R}^d$ , there exists a polytope  $\Sigma(\mathcal{A}) \subset \mathbb{R}^{|\mathcal{A}|}$ , the **secondary polytope** of  $\mathcal{A}$ , s.t.

1. Vertices of  $\Sigma(\mathcal{A})$  correspond bijectively to regular triangulations of  $\mathcal{A}$  via  $T \mapsto \varphi_T : (\varphi_T)_i = \sum_{a_i \in \text{Vert}(\sigma) : \sigma \in T} \text{Vol}(\sigma)$ ,  $i = 1, \dots, |\mathcal{A}|$ ,
2. Edges correspond to flips.

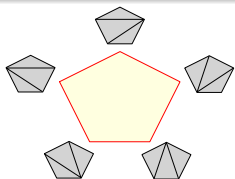


Secondary polytope of a pentagon.

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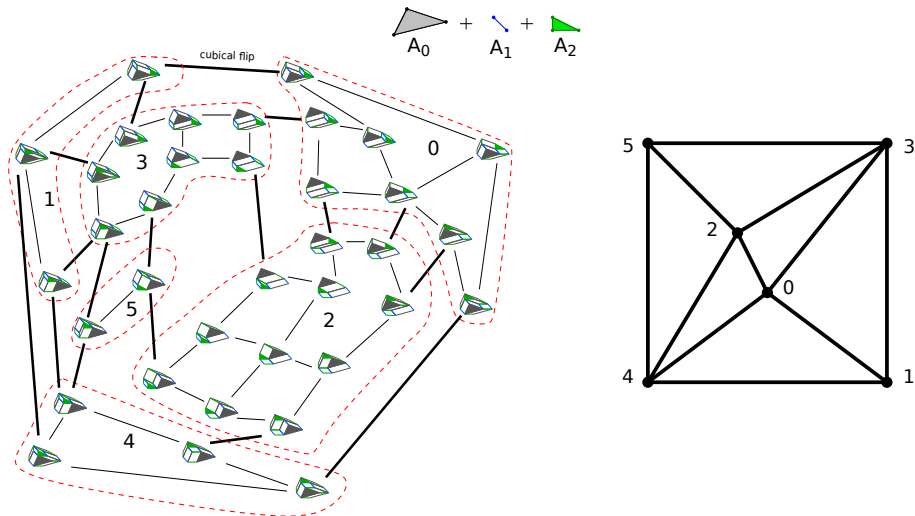
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Secondary polytope of a pentagon.

$N(\mathcal{R})$  is a Minkowski summand of  $\Sigma(\mathcal{A})$ .

# Example



(Top) Input polytopes. (Left) Graph of mixed subdivisions of  $A_0, A_1, A_2$  (equiv., triangulations of  $\mathcal{A}$ ): the 6 equivalence classes are dotted. (Right)  $N(\mathcal{R}) \subset \mathbb{R}^3$ .

Given  $n + 1$  polynomials in  $n$  variables with fixed supports  $A_i \subset \mathbb{Z}^n$ , efficiently compute  $\pi(N(\mathcal{R})) = II \subset \mathbb{R}^m$  for some orthogonal projection  $\pi$ .

## Idea

- Avoid computing the whole secondary polytope  $\Sigma(\mathcal{A})$ .
- Incrementally construct  $II$  using an oracle that given a direction produces a vertex of  $II$ , extremal wrt this direction.



## Interpolation of resultants

$$f_0 = a_2x^2 + a_1x + a_0, \quad f_1 = b_1x^2 + b_0$$

$N(\mathcal{R})$ : convex hull of  $(0, 2, 0, 1, 1)$ ,  $(0, 0, 2, 2, 0)$ ,  $(2, 0, 0, 0, 2)$ .

Contains 1 additional lattice point:  $(1, 0, 1, 1, 1)$ .

Possible monomials of  $\mathcal{R}$ :  $a_1^2b_1b_0$ ,  $a_0^2b_1^2$ ,  $a_2a_0b_1b_0$ ,  $a_2^2b_0^2$

Using the 4 monomials and the Horn-Kapranov parameterization of  $\mathcal{R}$ :

$$a_2 = (2t_1 + t_2)t_3^2t_4, \quad a_1 = (-2t_1 - 2t_2)t_3t_4, \quad a_0 = t_2t_4, \quad b_1 = -t_1t_3^2t_5, \quad b_0 = t_1t_5,$$

construct a  $4 \times 4$  matrix whose kernel vector  $(1, 1, -2, 1)$  contains their coefficients:

$$\mathcal{R} = a_1^2b_1b_0 + a_0^2b_1^2 - 2a_2a_0b_1b_0 + a_2^2b_0^2$$

## Implicitization of parametric (hyper)surfaces

$$(y_1, y_2, y_3) = (x_1 x_2, x_1 x_2^2, x_1^2).$$

$$f_0 := c_{00} - c_{01} x_1 x_2, \quad f_1 := c_{10} - c_{11} x_1 x_2^2, \quad f_2 := c_{20} - c_{21} x_1^2$$

$$\mathcal{R} = -c_{00}^4 c_{11}^2 c_{21} + c_{01}^4 c_{10}^2 c_{20}, \quad N(\mathcal{R}) = ((4, 0, 0, 2, 0, 1), (0, 4, 2, 0, 1, 0)) \subset \mathbb{R}^6$$

$$\text{Define } \pi : \pi(c_{00}, c_{01}, c_{10}, c_{11}, c_{20}, c_{21}) = (y_1, -1, y_2, -1, y_3, -1)$$

Then,  $\pi(\mathcal{R}) = -y_1^4 + y_2^2 y_3$  is the implicit equation and  $\Pi = \pi(N(\mathcal{R})) = ((4, 0, 0), (0, 2, 1))$  is the implicit polytope.

Note that here:  $\pi(N(\mathcal{R})) = N(\pi(\mathcal{R}))$

## Theory

- [Michiels & Verschelde \[DCG'99\]](#) define equivalence classes of mixed subdivisions mapping onto  $N(\mathcal{R})$  vertices.
- [Michiels & Cools \[DCG'00\]](#) describe all Minkowski summands of  $\Sigma(\mathcal{A})$  including  $N(\mathcal{R})$ .

## Algorithms

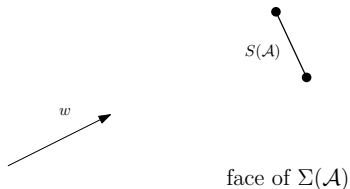
- [TOPCOM \[Rambau '02\]](#) computes all mixed subdivisions. Inefficient for resultant polytope computation.
- Tropical geometry [\[Sturmfels-Yu '08\]](#) leads to algorithms for the polytope of (specialized) resultant based on the `GFan` library [\[Jensen-Yu '11\]](#).

## Vertex $\Pi(\mathcal{A}, w)$

Input:  $\mathcal{A} \subset \mathbb{Z}^{2n}$ , direction  $w \in (\mathbb{R}^{|\mathcal{A}|})^\times$

Output: vertex  $\in \Pi$ , extremal wrt  $w$

- 1 use  $w$  as a lifting to construct regular subdivision  $S(\mathcal{A})$

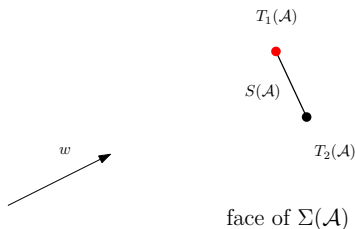


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- 2 refine  $S(\mathcal{A})$  into triangulation  $T(\mathcal{A})$

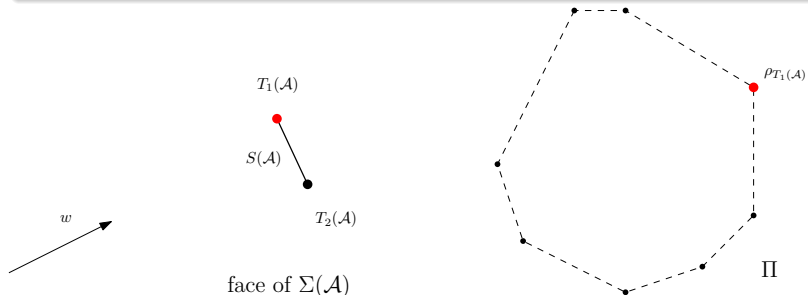


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- 1 use  $w$  as a lifting to construct regular subdivision  $S(\mathcal{A})$
- 2 refine  $S(\mathcal{A})$  into triangulation  $T(\mathcal{A})$
- 3 return  $\rho_{T(\mathcal{A})} \in \mathbb{N}^{|\mathcal{A}|}$

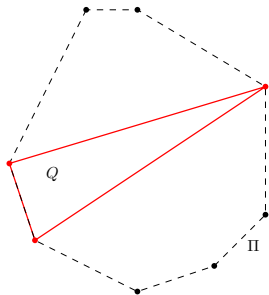


# Incremental Algorithm

Input:  $\mathcal{A}$

Output: H-rep.  $Q_H$ , V-rep.  $Q_V$  of  $Q = \Pi$

## 1 initialization step



initialization:

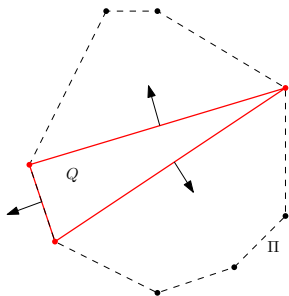
- $Q \subset \Pi$
- $\dim(Q) = \dim(\Pi)$

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- 1 initialization step
- 2 all hyperplanes of  $Q_H$  are **illegal**



2 kinds of hyperplanes of  $Q_H$  :

- **legal** if it supports a facet of  $\Pi$
- **illegal** otherwise

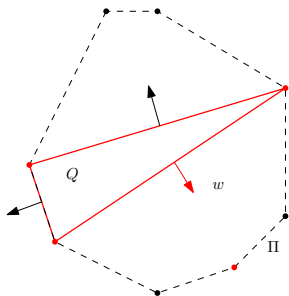


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- 3 while  $\exists$  illegal hyperplane  $H \subset Q_H$  with outer normal  $w$  do
  - compute  $v = \text{VertexII}(\mathcal{A}, w)$ ,  $Q_V \leftarrow Q_V \cup \{v\}$



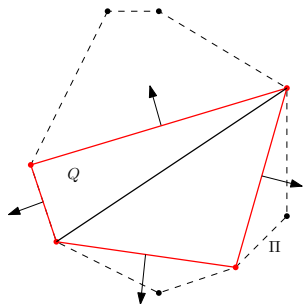
Extending an **illegal** facet

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  - compute  $v = \text{Vertex}\Pi(\mathcal{A}, w)$ ,  $Q_V \leftarrow Q_V \cup \{v\}$
  - if  $v \notin Q_V \cap H$  then  $Q_H \leftarrow \text{CH}(Q_V \cup \{v\})$  else  $H$  is **legal**



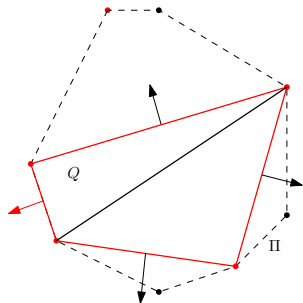
Extending an **illegal** facet

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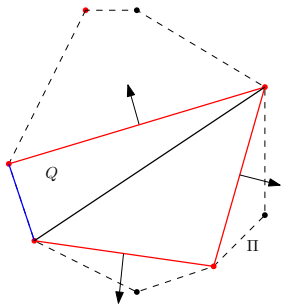
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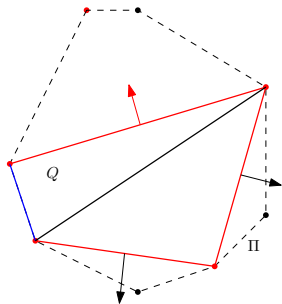
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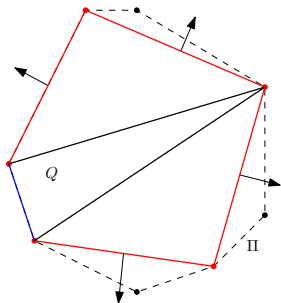


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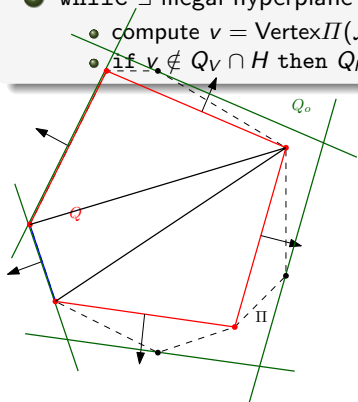
At any step,  $Q$  is an inner approximation of  $\Pi \dots$

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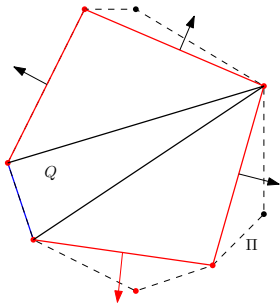
At any step,  $Q$  is an inner approximation of  $\Pi \dots$  from which we can compute an outer approximation  $Q_o$ .

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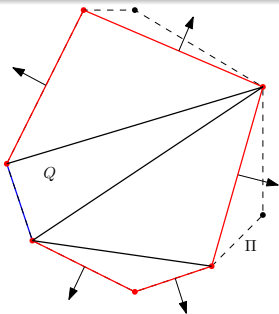


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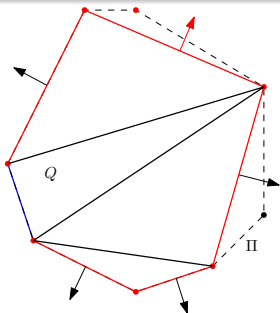


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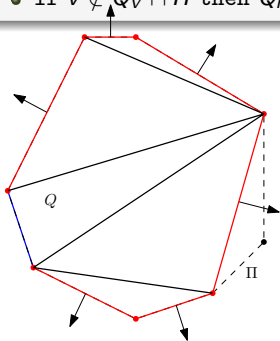


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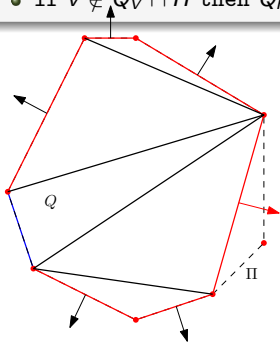


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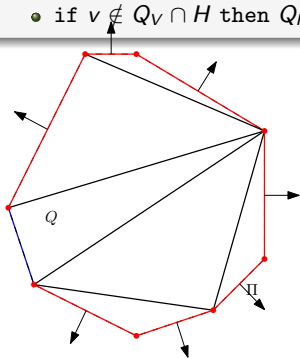


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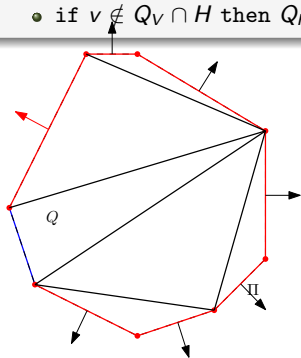


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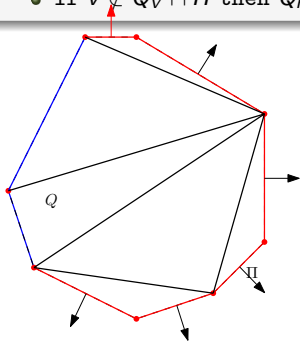


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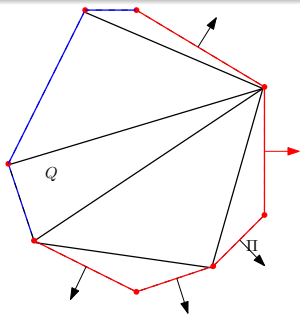


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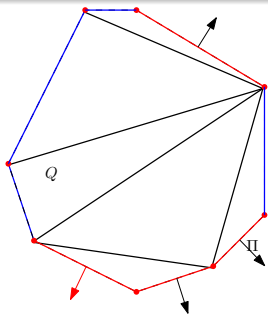


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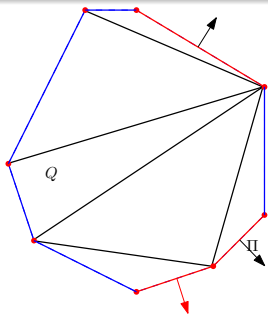


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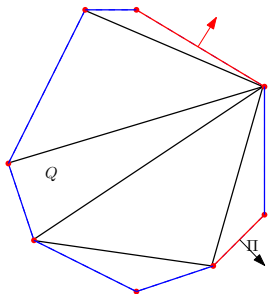


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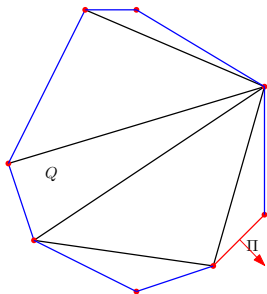


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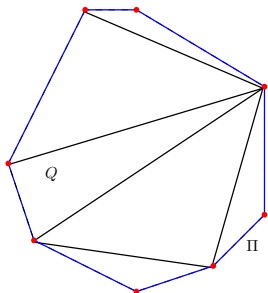


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## Theorem

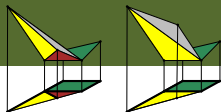
We compute the Vertex- and Halfspace-representations of  $\Pi$ , as well as a triangulation  $T$  of  $\Pi$ , in

$$O^*(m^5 |\text{vtx}(\Pi)| \cdot |T|^2),$$

where  $m = \dim \Pi$ , and  $|T|$  the number of full-dim faces of  $T$ .

## Elements of proof

- Most computation is done in dimension  $\leq m$ .
- At most  $\leq \text{vtx}(\Pi) + \text{fct}(\Pi)$  oracle calls
- Beneath-Beyond algorithm for converting V-rep. to H-rep. (bottleneck)



## Tools

C++, CGAL, triangulation [Boissonnat,Devillers,Hornus], extreme\_points\_d [Gärtner]

## Hashing of determinantal predicates

optimizing sequences of similar determinants

$$\tilde{\mathcal{A}} = \begin{array}{c|cccccccc|} 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 & w_9 \end{array}$$

- Laplace (cofactor) expansion wrt the last row + Hash minors
- If all needed minors computed, orientation =  $O(n^2)$ , volume =  $O(n)$

- approximate resultant polytopes ( $\dim(\Pi) \geq 7$ )
- preliminary results:

	m   $\mathcal{A}$	3	3	4	4	5	5
		input	200	490	20	30	17
exact	#vtx( $\Pi$ )	98	133	416	1296	1674	5093
	time	2.03	5.87	3.72	25.97	51.54	239.96
approx.	#vtx( $Q_{in}$ )	15	11	63	121	–	–
	vol( $Q_{in}$ )/vol( $\Pi$ )	0.96	0.95	0.93	0.94	–	–
	vol( $Q_{out}$ )/vol( $\Pi$ )	1.02	1.03	1.04	1.03	–	–
	time	0.15	0.22	0.37	1.42	> 10hr	> 10hr

- approximate volume computation [Lovsz-Vempala06]



## The code

- <http://respol.sourceforge.net>

## The paper

- <http://arxiv.org/abs/1108.5985v2> (to appear in SOCG'12)

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Thank You !