Semiclassical propagation of waves in chaotic media: An overview of recent results and challenges

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Quantum and classical dynamics

quantum evolution (or wave propagation) $\hbar \to 0$ classical evolution (Hamiltonian)

$\hbar \to 0 \leftrightarrow$ short-wavelength/high-frequency limit

Main question in semiclassics/quantum chaos:

What is the influence of dynamical properties of the classical system on the wave propagation for small $\hbar$?

- quantitative: computational tools: PDE $\to$ ODE
- qualitative: e.g. chaos and universality, integrability, bifurcations ....
- main (technical ?) problem: two limits, $\hbar \to 0$, $t \to \infty$

The Schrödinger equation

- $M = \mathbb{R}^n$ or (M, g) compact Riemannian manifold, e.g., a billiard.
- Schrödinger equation for particle in potential V on M:

$$\mathrm{i}\hbar\partial_t\psi(t) = -\frac{\hbar^2}{2}\Delta_g\psi(t) + V\psi(t)$$

- will choose initial conditions of the form

$$\psi(t=0) = \psi_0 = A\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$$

  where $A, S \in C^\infty(M)$ and S real valued.
- *oscillatory* initial conditions $\to$ hyperbolic problem (in the PDE sense), so we expect propagation.
- Unitary time evolution operator (propagator) $\mathcal{U}(t) : L^2(M) \to L^2(M)$:

$$\psi(t) = \mathcal{U}(t)\psi_0 \ .$$

Hamiltons equations

- phase space: T^*M, local coordinates $(p,q) \in \mathbb{R}^n \times U$, $U \subset \mathbb{R}^n$
- Hamilton function: Energy

$$H(p,q) = \frac{1}{2}p^2 + V(q)$$

energy shell: $\mathcal{E}_E := \{H(p,q) = E\}$ Liouville measure $\mu_E = C\delta(H(p,q) - E)\mathrm{d}^n p\mathrm{d}^n q$ (normalised if $\mathcal{E}_E$ is compact)

- Hamiltons equations:

$$\dot{p} = -\nabla_q H(p,q) \ , \quad \dot{q} = \nabla_p H(p,q) \ .$$

- Hamiltonian flow: $\Phi^t : T^*M \to T^*M$, $(p(t), q(t)) = \Phi^t(p_0, q_0)$ solution to Hamiltons equation with $p(0) = p_0, q(0) = q_0$.
- energy shell $\mathcal{E}_E$ and Liouville measure μ_E are invariant under the flow Φ^t.

Lagrangian states

$\psi = A e^{\frac{i}{\hbar} S}$ A amplitude S phase function

- Associated **Lagrangian submanifold**

$$\Lambda_S = \{(\nabla S(q), q), q \in \operatorname{supp} A\} \in T^* M$$

- $P(D, q)$, $D = -i\hbar\nabla$, (pseudo)differential operator

$$\langle \psi, P(D, q)\psi \rangle = \int_\Lambda P \, \mathrm{d}\nu_\psi + O(\hbar)$$

$\mathrm{d}\nu_\psi$ measure on Λ_S defined by ψ.

- up to **caustics**

$$\mathcal{U}(t)\psi = [T(t)D(t)A] e^{\frac{i}{\hbar} S(t)}$$

  - $\partial_t S + H(\nabla_q S, q) = 0$ hence $\Phi^t(\Lambda_S) = \Lambda_{S(t)}$
  - $\partial_t T(t) = -(\nabla S \cdot \nabla + \Delta S) T(t)$ unitary, **transport**
  - $i\hbar \partial_t D(t) = -\frac{\hbar}{2} T^*(t) \Delta T(t) D(t)$ unitary, **dispersion**, and if A is smooth $D(t)A = A + O_t(\hbar)$.

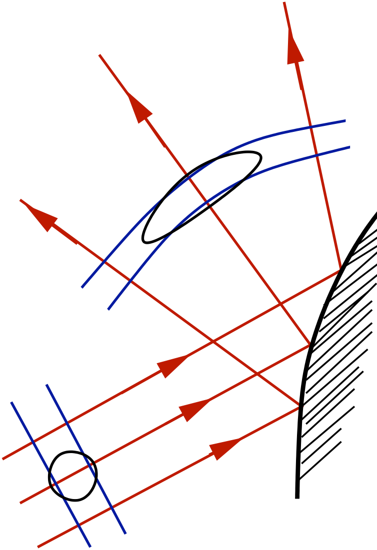
Semiclassical wave propagation

propagated wave:

$$\psi(t) \approx A(t)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t)}$$

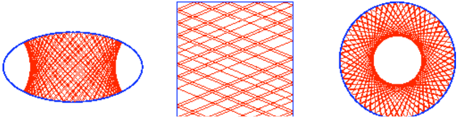
- $A(t)$ amplitude,
- $S(t)$ phase function
- **wave fronts**: $S(t,x) = const$

wavefronts and amplitude are transported along classical trajectories perpendicular to the wavefronts.

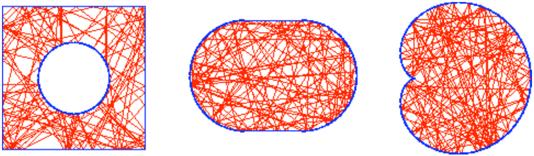


regular and irregular motion

regular (integrable) motion:



irregular (chaotic) motion:



- typically neighbouring trajectories diverge exponentially $\delta q(t) \sim \mathrm{e}^{\lambda t}$ (hyperbolicity)
- a typical trajectory fills the space with uniform density (ergodicity)

Balasz-Berry (79) random wave conjecture

If the classical system is *chaotic (hyperbolic)*, the wavepacket gets stretched at an exponential rate along the wavefronts, therefore several branches of the wavepacket start overlap and interfere with each other. We obtain

$$\psi(t) \sim \sum_{|\alpha| \leq \mathrm{e}^{\lambda t}} \psi_\alpha(t)$$

and the individual terms are expected to become *independent*.

- Expect:
  - equi-distribution on macroscopic scales $\gg \sqrt{\hbar}$ (mixing)
  - universal fluctuations (central limit theorem (CLT))
- randomisation of dynamical origin, in contrast to prevous random wave models in wave theory, e.g., Longuet-Higgins for water waves, or acoustic waves for complex systems.

Movies (by Arnd Bäcker)

Cardioid Billiard: in polar coordinates $r(\varphi) = 1 - \cos\varphi$, the billiard flow is chaotic.
Hamiltonian: Laplace operator $-\Delta$ with Dirichlet boundary conditions.
initial condition: $\psi_0(x) = \mathrm{e}^{-\alpha(x-x_0)^2/2}\mathrm{e}^{\frac{\mathrm{i}}{\hbar}kx}$, $\alpha > 0$, $k \in \mathbb{R}^2$.
We plot probability distribution in position space,

$$|\mathcal{U}(t)\psi_0(x)|^2$$

For observable $f(x)$, expected value at time t

$$\langle \mathcal{U}(t)\psi_0, f\,\mathcal{U}(t)\psi_0 \rangle = \int_M f(x)|\mathcal{U}(t)\psi_0|^2(x)\,\mathrm{d}x \ .$$

(some) History

- 1978-1979 Berman Zaslavsky, Balasz Berry Tabor Voros: log breaking time, Ehrenfest time $T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$ limit of validity of semiclassics?
- 1979 Berry Balasz - random wave conjecture: equidistribution and universal fluctuations.
- 1991 Tomsovic Heller: validity of semiclassics beyond Ehrenfest time.
- 2000 Bonechi DeBiévre: equidistribution of coherent states in cat map on Ehrenfest time scales.

Mixing and Universality

A dynamical system $(X, \phi^t, \mathrm{d}\mu)$ is *mixing* if for $a, \rho \in L^2(X)$ $(\int_X \rho \, \mathrm{d}\mu = 1)$

$$\lim_{t\to\infty} \int_X a \circ \phi^t \rho \, \mathrm{d}\mu = \int_X a \, \mathrm{d}\mu$$

Interpretation:

- a observable, ρ a state (i.e., a probability density), then $\int a \circ \phi^t \rho \, \mathrm{d}\mu$ expected value of a at time t: the system forgets where it came from - "Universality"
- Other manifestations of universality: If mixing is rapid enough a Central Limit Theorem (CLT) holds

$$\frac{1}{\sqrt{T}} \int_0^T a \circ \Phi^t \mathrm{d}t \quad \text{becomes normally distributed}$$

for $T \to \infty$ (if $\int_X a \, \mathrm{d}\mu = 0$).

Anosov flows

Definition

ϕ^t is called Anosov if for all $x \in X$ there is a splitting

$$T_x X = E^s(x) \oplus E^u(x) \oplus E^0(x)$$

such that $E^0(x)$ is spanned by the flow direction and there are constants $\lambda > 0, C$ such that

- $\|\mathrm{d}\phi^t u\| \leq C\mathrm{e}^{-\lambda t}\|u\|$ *for all $u \in E^s(x)$ and $t > 0$*
- $\|\mathrm{d}\phi^t u\| \leq C\mathrm{e}^{\lambda t}\|u\|$ *for all $u \in E^u(x)$ and $t < 0$*

Example: geodesic flow on compact manifold M.

- $X = S^*M$, $\mathrm{d}\mu$ Liouville measure on S^*M
- ϕ^t Hamiltonian flow generated by $H(x, \xi) = \frac{1}{2}|\xi|^2_{g(x)}$

Geodesic flow is Anosov if all sectional curvatures are negative.

exponential mixing

Theorem (Dolgopyat 98, Liverani 05)

Assume X is compact and contact (e.g. $X = \mathcal{E}_E$) and $\phi^t : X \to X$ is Anosov and volume preserving, then there exists a $\gamma > 0$ such that for all $a, \rho \in C^1(X)$, $\int \rho \, \mathrm{d}\mu = 1$,

$$\int a \circ \phi^t \rho \, \mathrm{d}\mu = \int a \, \mathrm{d}\mu + O(\|a\|_{C^1} \|\rho\|_{C^1} \mathrm{e}^{-\gamma|t|})$$

Localise initial conditions: $\rho_\varepsilon(x) := \frac{1}{\varepsilon^{2d-1}} \rho_0\big(\frac{d(x,x_0)}{\varepsilon}\big)$, then $\|\rho_\varepsilon\|_{C^1} \sim 1/\varepsilon^{2d}$, so

$$\lim_{\varepsilon \to 0} \int a \circ \phi^t \rho_\varepsilon \, \mathrm{d}\mu = \begin{cases} a\big(\phi^t(x_0)\big) & \text{for} \quad t \ll \frac{1}{\lambda} \ln \frac{1}{\varepsilon} \\ \int a \, \mathrm{d}\mu & \text{for} \quad t \gg \frac{1}{\gamma} \ln \frac{1}{\varepsilon} \end{cases}$$

transition from local to global behaviour. Later $\varepsilon \sim \sqrt{\hbar}$, uncertainty relation

Equidistribution in the Anosov case

Theorem (RS 05)

Assume the Hamiltonian flow of H is Anosov on Σ_E, then there exist constants $\Gamma, \gamma > 0$ such that for any $\psi = A\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$ with $\|\psi\|_{L^2} = 1$, $\Lambda_S \subset \mathcal{E}_E$ and Λ_S is transversal to the stable foliation (i.e. the wavefronts are expanding), and any $f \in C^\infty(M)$

$$\int_M f(x)|\mathcal{U}(t)\psi|^2(x)\mathrm{d}x = \frac{1}{|M|}\int_M f(x)\,\mathrm{d}x + O(\hbar \mathrm{e}^{\Gamma|t|}) + O(\mathrm{e}^{-\gamma|t|})$$

- Remainder small if $0 \ll t \ll \frac{1}{\Gamma}\ln\frac{1}{\hbar}$, *Ehrenfest time* $T_E := \frac{1}{\Gamma}\ln\frac{1}{\hbar}$.
- Similar universality as in the classical system (mixing).
- Result is valid for more general operators and systems.

proof strategy

Step 1: Semiclassics (microlocal analysis): $\mathrm{Op}[f] \approx f(x, \hbar D_x)$, Egorov's Theorem (Bouzouina, Robert 02):

$$\mathcal{U}^*(t)\,\mathrm{Op}[f]\mathcal{U}(t) = \mathrm{Op}[f \circ \phi^t] + O_{L^2}(\hbar \mathrm{e}^{\Gamma t})$$

$$\langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi \rangle = \int_\Lambda f \circ \phi^t |\tilde{A}|^2 + O(\hbar \mathrm{e}^{\Gamma |t|})$$

Step 2: mixing: If Λ is transversal to stable foliation (expanding)

$$\int_\Lambda f \circ \phi^t |\tilde{A}|^2 = \frac{1}{|M|} \int_M f \;\mathrm{d}x \int_M |A|^2 \;\mathrm{d}x + O(\mathrm{e}^{-\gamma t})$$

Idea goes back to Margulis.

Localized states

- coherent states: $\psi = A_\hbar \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}$, $A_\hbar(x) = \hbar^{-n/4} A_0\big(\hbar^{-1/2}(x-q)\big)$ A_0 smooth.
- concentrated around $(p = \nabla S(q), q) \in T^*M$, width $\sim \sqrt{\hbar}$.
- Propagation: $\mathcal{U}(t)\psi = [T(t)D(t)A_\hbar]\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(t)}$, but $D(t)A_\hbar - A_\hbar = O(1)$
  - $\mathrm{i}\hbar\partial_t D(t) = -\frac{\hbar^2}{2}\Delta(t)D(t)$, $\Delta(t) = T^*(t)\Delta T(t)$
  - main idea: freeze coefficients of $\Delta(t)$ at $x = q$: $\Delta_q(t)$ resulting $D_q(t)$ is simple and

$$D(t)A_\hbar = D_q(t)A_\hbar + O_t(\sqrt{\hbar})$$

Work in progress: Φ^t Anosov, $\Lambda_S \subset \mathcal{E}_E$, and Λ_S transversal to the stable foliation. Then for all $f \in C_0^\infty(T^*M)$ (and if $\|\psi\|_{L^2} = 1$)

$$\lim_{t\to\infty} \langle \mathcal{U}(t)\psi, \mathrm{Op}[f]\mathcal{U}(t)\psi\rangle = \begin{cases} f\big(\Phi^t(p,q)\big) & \text{if} \quad t < \frac{1}{2\lambda}\ln\frac{1}{\hbar} \\ \int_{S^*M} f \,\mathrm{d}\mu & \text{if} \quad \frac{1}{2\lambda}\ln\frac{1}{\hbar} \ll t \ll \frac{1}{\lambda}\ln\frac{1}{\hbar} \end{cases}$$

time scales

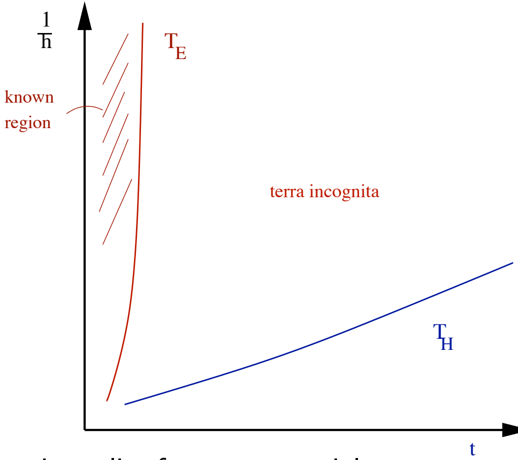
- Ehrenfest time

$$T_E \sim \frac{1}{\lambda} \ln \frac{1}{\hbar}$$

-exponential proliferation of orbits,
-small-scale oscillations

- Heisenberg time, time scale to resolve spectrum:

$$T_H \sim \frac{1}{\hbar^{d-1}}$$



Beyond Ehrenfest time: expect universality from exponential mixing and CLT

- effective averaging from uncertainty principle
- generic long orbits become dense and behave universal.

Outlook and Summary

- Semiclassical propagation of wave-packets is driven by the propagation of wave-fronts by the classical flow.
- Mixing of the classical flow in the Anosov case implies equidistribution in wave propagation.
- Similar results for integrable systems exist [RS 05].
- The main open problem is to go beyond Ehrenfest time, some recent progress in [RS 07] where the problem is reduced to *dispersive estimates* on $D(t)$ (on the hyperbolic plane)
- current extensions to complex potentials
- Recent work by Macia, Anantharaman & Macia, Anantharaman & Riviere on limits of

$$\int_{\mathbb{R}} \int_{M} f(t,x) |\mathcal{U}(t/\hbar)\psi|^2 \, \mathrm{d}x \mathrm{d}t$$

for $\hbar \to 0$.