Multi-Dimensional Gas Flow: Some Historical Perspectives

Tai-Ping Liu

Academia Sinica, Taiwan
Stanford University

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Euler equations in gas dynamics

$$\rho_t + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u}) = 0, \text{ continuity equation,}$$

$$(\rho \boldsymbol{u})_t + \nabla_{\boldsymbol{x}} \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u} + p(\rho) I) = 0, \text{ momentum equations.}$$

Potential flows

$$\boldsymbol{u} = \nabla_{\boldsymbol{x}} \phi.$$

Bernoulli equation

$$\phi_t + \frac{1}{2} |\nabla_{\boldsymbol{x}} \phi|^2 + \Pi(\rho) = A \text{ constant,}$$

$$\Pi'(\rho) = \frac{p'(\rho)}{\rho}, \ \sqrt{p'(\rho)} = c \text{ sound speed.}$$

Potential flow equation = Bernoulli equation + continuity equation

$$\phi_{tt} + 2\nabla_{\boldsymbol{x}}\phi \cdot \nabla_{\boldsymbol{x}}(\phi_t) + (\nabla_{\boldsymbol{x}}\phi)^t \nabla^2_{\boldsymbol{x}}\phi \nabla_{\boldsymbol{x}}\phi - c^2 \Delta\phi = 0.$$

Stationary potential flow equation

$$(\nabla_{\boldsymbol{x}}\phi)^t \nabla^2_{\boldsymbol{x}}\phi \nabla_{\boldsymbol{x}}\phi - c^2 \Delta\phi = 0,$$

Elliptic for subsonic flows, $|\nabla_{\boldsymbol{x}}\phi|^2 = \boldsymbol{u}| < c$,
Hyperbolic for supersonic flows, $|\nabla_{\boldsymbol{x}}\phi|^2 = \boldsymbol{u}| > c$.

Self-similarity variable $\xi = \boldsymbol{x}/t$.

$$\phi(\boldsymbol{x}, t) = t\psi(\xi),\ \chi(\xi) = \psi(\xi) - \frac{1}{2}|\xi|^2,$$

$$\nabla_{\xi}\psi = \nabla_{\boldsymbol{x}}\phi = \boldsymbol{u},\ \text{velocity.}$$

$$\nabla_{\xi}\chi = \boldsymbol{u} - \xi,\ \text{pseudo-velocity.}$$

Self-similar potential flow equation :

$$(\nabla_{\xi}\psi - \xi)^t \nabla_{\xi}^2\psi(\nabla_{\xi}\psi - \xi) - c^2\Delta\psi = 0,$$

$$(\nabla_{\xi}\psi - \xi)^t \nabla_{\xi}^2\psi(\nabla_{\xi}\psi - \xi) - c^2\Delta\psi = 2c^2 + |\nabla_{\xi}\chi|^2.$$

Elliptic for psudo-subsonic flows, $|\nabla_{\xi}\chi| = |\boldsymbol{u} - \xi| < c$,
Hyperbolic for psudo-supersonic flows, $|\boldsymbol{u} - \xi| > c$.

Two-dimensional potential flow
$\boldsymbol{x} = (x, y),\ \boldsymbol{\xi} = (\xi, \eta),\ \boldsymbol{u} = (u, v) = (\phi_x, \phi_y)$:

$$\phi_{tt} + 2\phi_x\phi_{xt} + 2\phi_y\phi_{yt} + [(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy},$$

Stationary potential flow equation

$$[(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0,$$

Elliptic for subsonic flows $(\phi_x)^2 + (\phi_y)^2 = u^2 + v^2 < c^2$.
Hyperbolic for supersonic flows $(\phi_x)^2 + (\phi_y)^2 = u^2 + v^2 > c^2$.

Two-dimensional potential flow

$$\boldsymbol{x} = (x, y),\ \boldsymbol{\xi} = (\xi, \eta),\ \boldsymbol{u} = (u, v) = (\phi_x, \phi_y).$$

Two-dimensional self-similar potential flow

$\phi(x, y, t) = t\psi(\xi, \eta),\ \chi(\xi, \eta) = \psi(\xi, \eta) - \frac{1}{2}(\xi^2 + \eta^2),\ \xi = \frac{x}{t};\ \eta = \frac{y}{t}$:

$$[c^2 - (\psi_\xi - \xi)^2]\psi_{\xi\xi} - 2(\psi_\xi - \xi)(\psi_\eta - \eta)\psi_{\xi\eta} + [c^2 - (\psi_\eta - \eta)^2]\psi_{\eta\eta} = 0,$$

$$(c^2 - (\chi_\xi)^2]\chi_{\xi\xi} - 2\chi_\xi\chi_\eta\chi_{\xi\eta} + (c^2 - (\chi_\eta)^2]\chi_{\eta\eta} = -2c^2 - |(\chi_\xi)^2 + (\chi_\eta^2|.$$

Elliptic for psudo-subsonic flows,
$(\chi_\xi)^2 + (\chi_\eta)^2 = (u - \xi)^2 + (v - \eta)^2 < c^2$.
Hyperbolic for psudo-supersonic flows, $(\chi_\xi)^2 + (\chi_\eta)^2 > c^2$.

Stationary equation

$$[(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0.$$

The stationary equation is usually posted as a boundary value problem with boundary data given at infinity. As with other situation in incompressible flows and elasticity, such a boundary value problem often does not have unique solutions. For compressible flows, this has been shown only for the quasi-one dimensional nozzle flows:
T.-P. Liu, Transonic gas flow in a duct of varying area, Arch. Rat. Mech. and Anal., 80 (1982), 1-18.
T.-P. Liu, Nonlinear stability and instability of transonic flows through a nozzle, Comm. Math. Phys., 83 (1982), 243-260.
T.-P. Liu, Nonlinear resonance for quasilinear hyperbolic equation, J. Math. Phys., 28 (1987), 2593-2602.

Self-similar potential equation

$$(c^2-(\chi_\xi)^2]\chi_{\xi\xi}-2\chi_\xi\chi_\eta\chi_{\xi\eta}+(c^2-(\chi_\eta)^2]\chi_{\eta\eta}=-2c^2-|(\chi_\xi)^2+(\chi_\eta)^2|.$$

The self-similar equation differs from the stationary equation in the additional lower order term $-2c^2-|(\chi_\xi)^2+(\chi_\eta)^2|$.
Physically, the boundary value problem for the self-similar equation, with boundary value also posted for $\xi^2+\eta^2$ at infinity, is equivalent to the initial value problem for the potential flow equation with self-similar initial data. Although it is difficult to prove, we expect the initial value problem for the potential flow equation to have unique solution.
Therefore the boundary value problem for the self-similar equation is expected to have a unique solution.

DISCUSSION ON THE EXISTENCE AND UNIQUENESS OR MULTIPLICITY OF SOLUTIONS OF THE AERODYNAMICAL EQUATIONS

Wednesday morning August 17, 1949

Participants: von Neumann, Liepmann, von Karman, Burgers, Heisenberg etc

von Neumann:

Occasionally the simplest hydrodynamical problems have several solutions, some of which are very difficult to exclude on mathematical grounds only. For instance, a very simple hydrodynamical problem is that of the supersonic flow of a gas through a concave corner, which obviously leads to the appearance of a shock wave. In general, there are two different solutions with shock waves, and it is perfectly well known from experimentation that only one of the two, the weaker shock wave, occurs in nature. But I think that all stability arguments to prove that it must be so, are of very dubious quality.

Liepmann: I would like to add a remark about the question of the two shock waves. I think that the experiments cannot be safely cited to settle whether only the solution with the weaker shock appears in nature, because the theoretical case refers to an infinite wall (or to the flow along the two sides of an infinite wedge), which case cannot be realized in practice. With the stronger one of the two shock waves you have subsonic flow behind the shock wave, which means that behind the shock wave you have a region where the theory of the elliptic differential equation applies and where the field is influenced by the boundary conditions at a finite or an infinite distance downstream. In the case of the other shock wave the velocity remains supersonic, so that you have conditions such as those obtained with hyperbolic equations. Thus one cannot exclude a priori that conditions downstream may influence the flow and thus may lead to a predilection for one type of shock wave about the other type.

von Karman: I would like to say something about this question of uniqueness of solutions. I dont think that there is any reason that if you put a problem in a form which has no physical meaning, there shall not be two solutions. And I think the case of stationary motion as such belongs to this category, because it can occur only as a limiting case. Any physical process starts from somewhere and goes to somewhere. In the case of the two shock waves, if instead of considering a stationary motion you consider an accelerated motion, you will first get a detached shock wave ahead of the obstacle (when the Mach number has just passed through unity). Then, with increasing velocity the solution will approach the correct solution for the steady case, I should think, without any difficulty. Such a case comes near to what you can actually realize in an experiment. Is that not correct?

von Neumann:
I may not have chosen that example which fits best to your argument. It has, of course, to be admitted that to postulate stationarity is to postulate a general trait of the solution one wants, which may hold only approximately in the physical situation that can actually be realized. However, it is not necessary to take the stationary flow through a corner. The following problem also has two solutions. If you take a plane shock which hits a wall and you consider the reflection of the shock from the wall, then under a wide variety of conditions (in fact, in most cases) there are two solutions. In this case stationarity has not been postulated.

von Karman: I only mean the following thing. I suppose we start from a certain state of rest of the gas, which must be a solution of our equations. Then we change the conditions gradually and follow the system step by step. I believe that in such a case you will always get a solution and only one solution. There is no proof that there is only one, but I believe it to be so. For, after all, a gas is a molecular system, which follows the general equations of classical mechanics. But if you take first an infinite cone, or an infinite wedgeboth of which are situations which can never be realizedand furthermore you ask for a stationary solution; in such a case there is no reason why there should be only one solution.

von Karman:
Since the equations are non-linear, you can often, without violating continuity, pass from one solution to another one by following an envelope, and in such a case you can scarcely find a mathematical reason why one solution should be preferred to the other. But if you start from an actually existing (observed) state and then determine the next phase, I believe you will find only one completely determined result.
Concerning Dr. von Neumanns example of the reflection of waves from a wall, I do not know the answer, but I believe that no case in which infinitely extending waves or walls are involved is really defined physically.

Burgers:
Dr. von Neumann mentioned a case of nonstationary theory where you have also two solutions: a shock wave hitting a wall. But in the picture you gave (Figure 3) the wall was infinite, so that here again one must ask: How does the situation arise, when you have an actual, finite wall? It may be that you could treat the problem for an actual situation, in which a shock wave travelling in unlimited space reaches the edge of a wall (see Figure 4), you might obtain a definite solution.

von Neumann:
In that case you assume that the state at the time t = 0 is given and you ask whether there is or is not a unique continuation of the solution at later times. The answer to this question in its full generality is not known; there seem to be a great many mathematical difficulties.

Multi-Dimensional Gas Flows

Stationary equation

$$[(\phi_x)^2 - c^2]\phi_{xx} + 2\phi_x\phi_y\phi_{xy} + [(\phi_y)^2 - c^2]\phi_{yy} = 0.$$

For an airfoil, as the upstream speed increases, a supersonic bubble will grow inside the subsonic region.
Self-similar equation

$$(c^2-(\chi_\xi)^2]\chi_{\xi\xi}-2\chi_\xi\chi_\eta\chi_{\xi\eta}+(c^2-(\chi_\eta)^2]\chi_{\eta\eta} = -2c^2-|(\chi_\xi)^2+(\chi_\eta)^2|.$$

A pseudo-supersonic bubble cannot grow inside a pseudo-subsonic region:
Volker Elling and T.-P. Liu
The ellipticity principle for self-similar potential flows., J. Hyperbolic Differ. Equ., 2(2005), no. 4, 909-917.

Suppose that, instead of gradual acceleration, a wedge is instantaneously accelerated to a supersonic speed. Then the solution is self-similar and it is shown that, for sufficiently pointed wedge, the solution contains a weak, supersonic shock attached to the edge of the wedge.

Volker Elling and T.-P. Liu

Supersonic flow onto a solid wedge., Comm. Pure Appl. Math., 61(2008), no. 10, 1347-1448.

Proof:

Global method of Ellipticity Principle and Leray-Schauder Degree.

Heisenberg:
I have one question in connection with these applications of the hydrodynamical equations. Should one assume from the beginning that these equations actually could be used to such a large extent? If we take the case of the gas expanding into a vacuum, the density at the front is so low that the mean free path becomes larger than the distance to the assumed front. Should one not start from the kinetic picture and say that at the front the molecules will sort themselves out according to their velocities?

Heisenberg:
Then the physical front would be formed by a selection of those molecules which had the highest velocities and did not suffer a collision for a long time. One should expect that there, especially, we have a velocity distribution different from the normal one, and therefore we should not apply the ordinary concepts like temperature and so on. I do not know how big the actual difference is, but I have tried to estimate it. One feels at least that there is a rather large region in which ordinary hydrodynamics cannot be applied, simply because the concepts of temperature and so on would be rather useless.

von Neumann:
Therefore, while it is certainly not rigorously true, dont you think it is sensible, first of all, to apply hydrodynamic theory, and get a solution? If you then discuss in what portions of the field the mean free path is small compared to the distances over which all essential changes occur (one of the most important portions is that where the distance from the boundary is small), it is reasonable to assume that the hydrodynamical equations may at least be used in such regions. When one has to deal with the boundary regions, the MaxwellBoltzmann theory should be called upon.

von Neumann:
Now what I have to say is that if one accepts this, and if one estimates how large these extraordinary regions are, in the cases which are of interest in the present context, they turn out to be fairly small. Properly speaking, in the case of the Riemann expansion into vacuum, the region where you have to be careful is quite large but it involves very little substance and very little energy. Hence, in many cases, the correction of the hydrodynamical solution in that region need not be discussed.

Heisenberg:
I certainly agree chiefly with what you say. I only would like to observe that the failures of hydrodynamical solutions determine the boundary conditions. The boundary conditions react back on the solutions of the hydrodynamic equations, and since these boundary conditions cannot be determined from hydrodynamics and require a detailed study of molecular processes, the two things are interconnected. With you, I believe that on the whole we can talk about hydrodynamical equations and their solutions, but the selection of the solutions to be used depends on the boundary conditions and to this extent we get these non-hydrodynamical parts of the field into our problem.

von Neumann:
The boundary layer theory for a fluid of low viscosity certainly furnishes a monumental warning. The naive and yet prima facie seemingly reasonable procedure would be to apply the ordinary equations of the ideal fluid and then to expect that viscosity will somehow take care of itself in a narrow region along the wall. We have learned that this procedure may lead to great errors; a complete theory of the boundary layer may give you completely different conditions also for the flow in the bulk of the field. It is possible that the same discipline will be necessary for the boundary with a vacuum. All I would like to say now is that there is yet no evidence for this.

Construction of Regular Reflection off a ramp:
Gui-Qiang Chen and Mikhail Feldman
Global Solutions of Shock Reflection by Large-Angle Wedges for Potential Flow, Annals of Mathematics, pp 108.
Optimal regularity of $C^{1,1}$ of the solution near pseudo-sonic circle:
Myoungjean Bae, Gui-Qiang Chen and Mikhail Feldman
Regularity of solutions to regular shock reflection for potential flow, Invent. Math, Vol. 505 No. 3 (2009) 505-543.

The transition criteria for various shock reflection patterns depend on the boundary condition and effective boundary. There are regular, Mach, complex Mach, etc reflections. There are sonic, geometric and other transition critetria.
The boundary condition and effective boundary for the compressible Euler equations are obtained through the coupling of the Knudsen and boundary layers with the Euler flows. The types of Knudsen and boundary layers depend on the physical scenario under consideration.