Invariant Manifolds for Steady Boltzmann Equation

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Boundary relation for some dissipative systems

Goal: Explicit Boundary Relation.

Boundary relation $\Rightarrow$ well-posed boundary conditions, Green's function for initial-boundary value problem.

Three examples:

Heat equation

$$u_t = u_{xx}.$$

Navier-Stokes

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Dissipative wave equations.

$$\begin{cases} u_t + u_x + v_y = \Delta u, \\ v_t - v_x + u_y = \Delta v. \end{cases}$$

Boundary relation for some dissipative systems

Heat equation

$$\begin{cases} u_t = u_{xx}, \ x, t \geq 0, \\ u(x,0) = 0, \ x \geq 0. \end{cases}$$

Boundary values:

$$\begin{cases} u^0(t) \equiv u(0,t), \ \text{Dirichlet boundary value}, \\ u_x^0(t) \equiv u_x(0,t), \ \text{Neumann boundary value}. \end{cases}$$

Laplace transforms

$$U(x,s) \equiv \int_0^\infty e^{-st} u(x,t)dt, \ \hat{U}(\xi,s) \equiv \int_0^\infty e^{-\xi x} U(x,s)dx,$$

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Heat equation
Laplace transforms

$$sU = U_{xx},\ (s-\xi^2)\hat{U} = -U_x^0 - \xi U^0.$$

Characteristic polynomial

$$p(\xi,s) \equiv \xi^2 - s = (\xi - \lambda_1(s))(\xi - \lambda_2(s)),$$

roots

$$\lambda_1(s) = -\sqrt{s} < 0 < \lambda_2(s) = \sqrt{s}.$$

We have

$$\hat{U} = \frac{U_x^0 + \xi U^0}{p_H(\xi,s)} = \frac{U_x^0 + \xi U^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}.$$

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Heat equation

Inversion of the Laplace transform, first in x:

$$U(x,s) = \frac{1}{2\pi i}\int_{-i\infty+D}^{i\infty+D} e^{\xi x}\frac{U_x^0 + \xi U^0}{(\xi-\lambda_1(s))(\xi-\lambda_2(s))}d\xi$$

in terms of the residues $Res(\xi = \pm\sqrt{s})$:

$$U(x,s) = e^{\lambda_1(s)x}\frac{U_x^0 + \lambda_1(s)U^0}{\lambda_1(s)-\lambda_2(s)} + e^{\lambda_2(s)x}\frac{U_x^0 + \lambda_2(s)U^0}{\lambda_2(s)-\lambda_1(s)}.$$

$e^{\lambda_2(s)x} = e^{\sqrt{s}x}$: unstable mode as $x \to \infty$. This yields Dirichlet-Neumann relation in the transformed variables:

$$U_x^0 + \lambda_2(s)U^0 = U_x^0 + \sqrt{s}U^0 = 0.$$

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Heat equation

Inverting the Laplace transform in s of

$$U_x^0 + \sqrt{s}U^0 = 0.$$

Instead of inverting $\sqrt{s}$, we will invert $\frac{\sqrt{s}}{s} = \frac{1}{\sqrt{s}}$. by the usual inversion formula of the Laplace transform:

$$L(t) \equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\frac{1}{\sqrt{s}}ds = \frac{1}{2\pi\sqrt{t}}\int_{-\infty}^{\infty} e^{is}\frac{1}{\sqrt{is}}ds = \frac{1}{\sqrt{\pi t}}.$$

Inversion of the Laplace transform: Division in s corresponds to differentiation in t. Dirichlet-Neumann relation:

$$u_x(0,t) = \frac{\partial}{\partial t}\int_0^t \frac{1}{\sqrt{\pi(t-s)}}u(0,s)ds.$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Zero initial value $(\rho, v)(x, 0) = 0, \ x > 0.$

Laplace transforms

$$\begin{cases} sP + V_x = 0, \\ sV + P_x = V_{xx}. \end{cases}$$

$$\begin{cases} s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} + P^0 - V^0_x - \xi V^0. \end{cases}$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\begin{cases} sP^0 + V^0_x = 0, \\ s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} - (\frac{1}{s} + 1)V^0_x - \xi V^0. \end{cases}$$

$\Rightarrow$

$$\hat{V} = \frac{\xi V^0 + V^0_x}{p(\xi, s)} = \frac{\xi V^0 + V^0_x}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}$$

Characteristic polynomial

$$p(\xi, s) \equiv \xi^2 - \frac{s^2}{(s+1)^2} = (\xi - \lambda_1(s))(\xi - \lambda_2(s)),$$

roots

$$\lambda_1(s) = -\frac{s}{\sqrt{s+1}} < 0 < \lambda_2(s) = \frac{s}{\sqrt{s+1}},.$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\hat{V} = \frac{\xi V^0 + V_x^0}{p(\xi, s)} = \frac{\xi V^0 + V_x^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}$$

inversion of the Laplace transform in x:

$$V(x,s) = e^{\lambda_1(s)x}\frac{\lambda_1(s)V^0 + V_x^0}{(\lambda_1(s) - \lambda_2(s))} + e^{\lambda_2(s)x}\frac{\lambda_2(s)V^0 + V_x^0}{(\lambda_2(s) - \lambda_1(s))}.$$

$e^{\lambda_2(s)x} = e^{sx/\sqrt{s+1}}$ unstable mode.

Dirichlet-Neumann relation in the transformed variables:

$$\lambda_2(s)V^0 + V_x^0 = \frac{s}{\sqrt{s+1}}V^0 + V_x^0 = 0.$$

Boundary relation for some dissipative systems

Navier-Stokes equations
Dirichlet-Neumann relation in the transformed variables:

$$\lambda_2(s)V^0 + V_x^0 = \frac{s}{\sqrt{s+1}}V^0 + V_x^0 = 0.$$

inverse Laplace transform:

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{e^{st}}{\sqrt{s+1}}ds = \frac{1}{2\pi\sqrt{t}}e^{-t}\int_{-\infty}^{\infty}\frac{e^{iz}}{\sqrt{iz}}dz = \frac{e^{-t}}{\sqrt{\pi t}}$$

Dirichlet-Neumann relation:

$$v_x(0,t) = \frac{d}{dt}[\frac{e^{-t}}{\sqrt{\pi t}} \star v(0,t)] = \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}}v(0,\tau)d\tau.$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Zero initial value $(\rho, v)(x, 0) = 0,\ x > 0.$
Dirichlet-Neumann relation:

$$v_x(0,t) = \frac{d}{dt}[\frac{e^{-t}}{\sqrt{\pi t}} \star v(0,t)] = \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}} v(0,\tau) d\tau.$$

Recover $\rho(0,t)$ from the continuity equation $\rho_t + v_x = 0$:

$$\rho(0,t) = \int_0^t v_x(0,s) ds = \int_0^t \int_0^s \frac{e^{-(s-\tau)}}{\sqrt{\pi(s-\tau)}} v(0,\tau) d\tau ds..$$

Therefore, only one boundary condition $v(0,t)$ for well-posedness.

Boundary relation for some dissipative systems

Remark: Convective Navier-Stokes equations

For the situation of evaporation $\Gamma > 0$, or condensation $\Gamma < 0$, the well-posed boundary conditions are different from each other and also from the above case of solid boundary:

$$\begin{cases} \rho_t + \Gamma \rho_x + v_x = 0, \\ v_t + +\Gamma v_x + \rho_x = v_{xx}. \end{cases}$$

The study of the boundary relations naturally yields the well-posed boundary conditions.

Wave equations

$$u_{tt} = u_{xx} + u_{yy}$$

can be written as a system of two first order equations

$$\begin{cases} u_t + u_x + v_y = 0, \\ v_t - v_x + u_y = 0, \end{cases} \qquad v \equiv \int^y -(u_t + u_x)dy.$$

Dissipative wave equations

$$\begin{cases} u_t + u_x + v_y = \Delta u \\ v_t - v_x + u_y = \Delta v \end{cases}$$

Boundary relation for some dissipative systems

Dissipative wave equations
Laplace (t)-Fourier (y) $(U, V)(x, \xi, s)$; Laplace (x)
$(\hat{U}, \hat{V})(\eta, \xi, s)$:

$$\begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} (1-\xi)U^0 - U_x^0 \\ (-1-\xi)V^0 - V_x^0 \end{pmatrix},$$

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \frac{1}{p(\xi,\eta,s)} \begin{pmatrix} s-\xi-\xi^2+\eta^2 & -i\eta \\ -i\eta & s+\xi-\xi^2+\eta^2 \end{pmatrix} \begin{pmatrix} (1-\xi)U^0 - U \\ (-1-\xi)V^0 - \end{pmatrix}$$

Characteristic polynomial $p(\xi, \eta, s)$, in ξ variable,

$$p(\xi,\eta,s) = det \begin{pmatrix} s+\xi-\xi^2+\eta^2 & i\eta \\ i\eta & s-\xi-\xi^2+\eta^2 \end{pmatrix}$$
$$= s^2+\eta^2+2s\eta^2+\eta^4-(1+2s+2\eta^2)\xi^2+\xi^4$$

Boundary relation for some dissipative systems

Dissipative wave equations

$$p(\xi, \eta, s) = s^2 + \eta^2 + 2s\eta^2 + \eta^4 - (1 + 2s + 2\eta^2)\xi^2 + \xi^4$$

has four roots, $\lambda_1,\ \lambda_2$ have positive real roots and correspond to unstable modes:

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \equiv \left\{ \sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1+4s} + \eta^2},\ \sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta^2}, \right.$$
$$\left. -\sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1+4s} + \eta^2},\ -\sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1+4s} + \eta^2} \right\}. \tag{1}$$

Boundary relation for some dissipative systems

Dissipative wave equations
Dirichlet-Neumann relation in the transformed variables

$$U_x^0 = \frac{1}{2}(1-\alpha-\beta)\,U^0 + \frac{i(1+\alpha-\beta)\left(1-2\alpha+\alpha^2-\beta^2\right)}{4\eta}\,V^0,$$

$$\begin{aligned} V_x^0 = {} & \frac{1}{8(1+4s)\eta^3} i(\alpha-\beta)\left(-\alpha+\alpha^2-\beta(1+\beta)\right) \\ & \left(1+4s+\beta^2-2\beta^3+\alpha^2(-1+2\beta)\right) \\ & \cdot[-1+\beta^2-2s(2+\alpha+\beta)+\alpha^2(1+2\beta)-2\eta^2-2\beta\eta^2 \\ & +2\alpha\left(\beta+\beta^2-\eta^2\right)]U^0 + \frac{1}{2}(-1-\alpha-\beta)\,V^0, \end{aligned}$$

Boundary relation for some dissipative systems

Dissipative wave equations The Dirichlet-Neumann relations in the transformed variables contain expressions which are polynomials in α and β where

$$\begin{cases} \alpha = \lambda_2 \equiv \sqrt{(\frac{1}{2} - \sqrt{s+1/4})^2 + \eta^2}, \\ \beta = \lambda_1 \equiv \sqrt{(\frac{1}{2} + \sqrt{s+1/4})^2 + \eta^2}. \end{cases} \tag{2}$$

As the functions α and β do not decay to zero for $s \in i\mathbb{R}$ as $|s| \to \infty$, instead we will invert the transforms for

$$\frac{\partial \alpha}{\partial s} = \left[\sqrt{\frac{1}{4} + s} - \frac{1}{2}\right] \Big/ \left[2\sqrt{\frac{1}{4} + s}\sqrt{\left(\frac{1}{2} - \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2}\right],$$

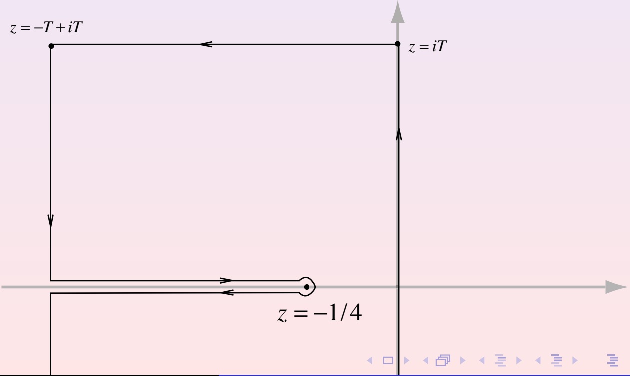
$$\frac{\partial \beta}{\partial s} = \left[\sqrt{\frac{1}{4} + s} + \frac{1}{2}\right] \Big/ \left[2\sqrt{\frac{1}{4} + s}\sqrt{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2}\right].$$

Boundary relation for some dissipative systems

Dissipative wave equations The Bromwich's integral of β_s for the inverse Laplace transform yielding exponential decay due to spectral gap, the real part of β_s is less than $-1/4$:

$$\frac{1}{2\pi i}\int_{Re(s)=0}\left(\frac{\partial\beta}{\partial s}\right)e^{st}ds = O(1)\frac{e^{-t/4}}{\sqrt{t}(1+|\eta|)}. \qquad (3)$$

using the contour integral over the path illustrated in Figure 1:



Boundary relation for some dissipative systems

Dissipative wave equations α_s, which can be arbitrarily close to zero as $\eta|$ varies. An entirely different, more sophisticated thinking is needed for computing the Bromwich integral,

$$\frac{1}{2\pi i}\int_{Re(s)=0}\frac{\partial\alpha}{\partial s}e^{st}ds=\frac{1}{2\pi i}\int_{Re(s)=0}\frac{1-\frac{1}{2\sqrt{\frac{1}{4}+s}}}{2\sqrt{\frac{1}{2}+s-\sqrt{\frac{1}{4}+s}+\eta^2}}e^{st}ds. \tag{4}$$

The function α_s is analytic in the cut domain

$$\mathbb{C}\setminus\left(\{z=-x^2\pm ix : x\geq|\eta|\}\cup(-\infty,-1/4]\right).$$

To compute the Bromwich integral we integrate α_s along the boundary of this domain, the contours in Figure 2:

Dissipative wave equations

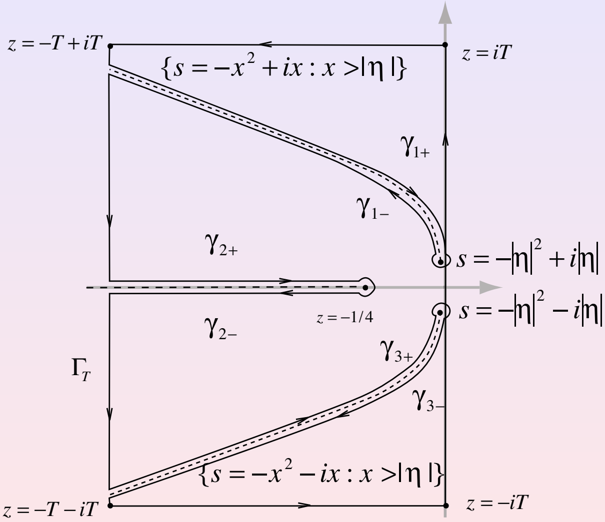


Figure 2. The contour of the path integral of Γ_T.

Boundary relation for some dissipative systems

Dissipative wave equations

The integral along $\gamma_{2\pm}$ decays exponentially in t due to the spectral gap.

The integrals along $\gamma_{1\pm}$ is characteristic in the sense that α_s can be arbitrarily close to the origin as η varies. We set

$$\sqrt{1/2 + s - \sqrt{1/4 + s} + \eta^2} = \pm i\tau$$

so that $-\infty < \tau < \infty$ along $\gamma_{1\pm}$.

Similarly, along $\gamma_{3\pm}$, set

$$\sqrt{1/2 + s - \sqrt{1/4 + s} + \eta^2} = i\tau.$$

We will integrate in τ instead of s. By long computations, we have

Dissipative wave equations

$$\int_{\gamma_{1+}+\gamma_{1-}} \frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}} e^{st}\,ds$$

$$= 2i\int_0^\infty e^{-t\left(\tau^2+\eta^2\right)}\left(\cos\left(t\sqrt{\tau^2+\eta^2}\right)+i\sin\left(t\sqrt{\tau^2+\eta^2}\right)\right)d\tau;$$

$$\int_{\gamma_{3+}+\gamma_{3-}} \frac{-\frac{1}{2}+\sqrt{\frac{1}{4}+s}}{2\sqrt{\frac{1}{4}+s}\sqrt{\left(-\frac{1}{2}+\sqrt{\frac{1}{4}+s}\right)^2+\eta^2}} e^{st}\,ds$$

$$= -2\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)}\left(i\cos\left(t\sqrt{\eta^2+\tau^2}\right)+\sin\left(t\sqrt{\eta^2+\tau^2}\right)\right)d\tau.$$

Boundary relation for some dissipative systems

Dissipative wave equations

The integral along $\gamma_{1\pm}$ and $\gamma_{3\pm}$ is inverted with respect to the Fourier variable η to yield the expression:

$$\mathbf{W}_2(y,t) \equiv -\frac{1}{\pi^2}\int_{\mathbb{R}^2} e^{iy\eta}\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)}\cos\left(t\sqrt{\eta^2+\tau^2}\right)d\tau d\eta.$$

This function can be identified with the convolution of the 2-dimensional heat kernel with the solution u of the following initial value problem of a wave equation at time t:

$$\begin{cases} \left(\dfrac{\partial^2}{\partial t'^2}-\dfrac{\partial^2}{\partial y'^2}-\dfrac{\partial^2}{\partial z'^2}\right)u(y',z',t')=0, \\ u(y',z',0)=-\dfrac{1}{\pi^2}\delta(y')\delta(z'), \\ u_{t'}(y',z',0)=0. \end{cases}$$

Boundary relation for some dissipative systems

Dissipative wave equations

The Green's function u for the 2-dimensional wave equation has been constructed by the Hadamard's descending method in terms of the Kirchhoff formula for 3-dimensional wave equation. The convolution with the heat kernel becomes:

$$
\begin{aligned}
&-\frac{1}{\pi^2}\int_{\mathbb{R}} e^{iy\eta}\int_0^\infty e^{-t\left(\eta^2+\tau^2\right)}\cos\left(t\sqrt{\eta^2+\tau^2}\right)d\tau d\eta \\
&\qquad = -\frac{1}{\pi^2}\,\partial_t'\left[\frac{1}{4\pi t'}\iint_{|\vec{y}-\vec{z}|=t'} k(\vec{z},t)dS_{\vec{y}}\right]\Bigg|_{t'=t},
\end{aligned}
$$

where

$$
\begin{cases}
\vec{y} = (y,0,0)\in\mathbb{R}^3, \\
\vec{z} = (z^1,z^2,z^3)\in\mathbb{R}^3, \\
k(\vec{\boldsymbol{z}},t) \equiv \dfrac{e^{-\frac{|z^1|^2+|z^2|^2}{4t}}}{4\pi t}.
\end{cases}
$$

Dissipative wave equations
Conclusion

The boundary relation for the dissipative wave equation is expressed in terms of convolution of local (exponentially decaying terms) operator with Hadamard-type operator.