

Invariant Manifolds for Steady Boltzmann Equation

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Crete, October 2011

Boundary relation for some dissipative systems

Goal: Explicit Boundary Relation.

Boundary relation \Rightarrow **well-posed boundary conditions,**
Green's function for initial-boundary value problem.

Three examples:

Heat equation

$$u_t = u_{xx}.$$

Navier-Stokes

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Dissipative wave equations.

$$\begin{cases} u_t + u_x + v_y = \Delta u, \\ v_t - v_x + u_y = \Delta v. \end{cases}$$

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Heat equation

$$\begin{cases} u_t = u_{xx}, & x, t \geq 0, \\ u(x, 0) = 0, & x \geq 0. \end{cases}$$

Boundary values:

$$\begin{cases} u^0(t) \equiv u(0, t), & \text{Dirichlet boundary value,} \\ u_x^0(t) \equiv u_x(0, t), & \text{Neumann boundary value.} \end{cases}$$

Laplace transforms

$$U(x, s) \equiv \int_0^{\infty} e^{-st} u(x, t) dt, \quad \hat{U}(\xi, s) \equiv \int_0^{\infty} e^{-\xi x} U(x, s) dx,$$

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Heat equation

Laplace transforms

$$sU = U_{xx}, \quad (s - \xi^2)\hat{U} = -U_x^0 - \xi U^0.$$

Characteristic polynomial

$$\rho(\xi, s) \equiv \xi^2 - s = (\xi - \lambda_1(s))(\xi - \lambda_2(s)),$$

roots

$$\lambda_1(s) = -\sqrt{s} < 0 < \lambda_2(s) = \sqrt{s}.$$

We have

$$\hat{U} = \frac{U_x^0 + \xi U^0}{\rho_H(\xi, s)} = \frac{U_x^0 + \xi U^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}.$$

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Heat equation

Inversion of the Laplace transform, first in x :

$$U(x, s) = \frac{1}{2\pi i} \int_{-i\infty+D}^{i\infty+D} e^{\xi x} \frac{U_x^0 + \xi U^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))} d\xi$$

in terms of the residues $\text{Res}(\xi = \pm\sqrt{s})$:

$$U(x, s) = e^{\lambda_1(s)x} \frac{U_x^0 + \lambda_1(s)U^0}{\lambda_1(s) - \lambda_2(s)} + e^{\lambda_2(s)x} \frac{U_x^0 + \lambda_2(s)U^0}{\lambda_2(s) - \lambda_1(s)}.$$

$e^{\lambda_2(s)x} = e^{\sqrt{s}x}$: **unstable mode** as $x \rightarrow \infty$. This yields **Dirichlet-Neumann relation** in the transformed variables:

$$U_x^0 + \lambda_2(s)U^0 = U_x^0 + \sqrt{s}U^0 = 0.$$

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Heat equation

Inverting the Laplace transform in s of

$$U_x^0 + \sqrt{s}U^0 = 0.$$

Instead of inverting \sqrt{s} , we will invert $\frac{\sqrt{s}}{s} = \frac{1}{\sqrt{s}}$ by the usual inversion formula of the Laplace transform:

$$L(t) \equiv \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{1}{\sqrt{s}} ds = \frac{1}{2\pi\sqrt{t}} \int_{-\infty}^{\infty} e^{is} \frac{1}{\sqrt{is}} ds = \frac{1}{\sqrt{\pi t}}.$$

Inversion of the Laplace transform: Division in s corresponds to differentiation in t . **Dirichlet-Neumann relation:**

$$u_x(0, t) = \frac{\partial}{\partial t} \int_0^t \frac{1}{\sqrt{\pi(t-s)}} u(0, s) ds.$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Zero initial value $(\rho, v)(x, 0) = 0, x > 0$.

Laplace transforms

$$\begin{cases} sP + V_x = 0, \\ sV + P_x = V_{xx}. \end{cases}$$

$$\begin{cases} s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} + P^0 - V_x^0 - \xi V^0. \end{cases}$$

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Navier-Stokes equations

$$\begin{cases} sP^0 + V_x^0 = 0, \\ s\hat{P} + \xi\hat{V} = V^0, \\ s\hat{V} + \xi\hat{P} = \xi^2\hat{V} - (\frac{1}{s} + 1)V_x^0 - \xi V^0. \end{cases}$$

\Rightarrow

$$\hat{V} = \frac{\xi V^0 + V_x^0}{p(\xi, s)} = \frac{\xi V^0 + V_x^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}$$

Characteristic polynomial

$$p(\xi, s) \equiv \xi^2 - \frac{s^2}{(s+1)^2} = (\xi - \lambda_1(s))(\xi - \lambda_2(s)),$$

roots

$$\lambda_1(s) = -\frac{s}{\sqrt{s+1}} < 0 < \lambda_2(s) = \frac{s}{\sqrt{s+1}},$$

Boundary relation for some dissipative systems

Navier-Stokes equations

$$\hat{V} = \frac{\xi V^0 + V_x^0}{\rho(\xi, s)} = \frac{\xi V^0 + V_x^0}{(\xi - \lambda_1(s))(\xi - \lambda_2(s))}$$

inversion of the Laplace transform in x :

$$V(x, s) = e^{\lambda_1(s)x} \frac{\lambda_1(s) V^0 + V_x^0}{(\lambda_1(s) - \lambda_2(s))} + e^{\lambda_2(s)x} \frac{\lambda_2(s) V^0 + V_x^0}{(\lambda_2(s) - \lambda_1(s))}.$$

$e^{\lambda_2(s)x} = e^{sx/\sqrt{s+1}}$ unstable mode.

Dirichlet-Neumann relation in the transformed variables:

$$\lambda_2(s) V^0 + V_x^0 = \frac{s}{\sqrt{s+1}} V^0 + V_x^0 = 0.$$

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Navier-Stokes equations

Dirichlet-Neumann relation in the transformed variables:

$$\lambda_2(s)V^0 + V_x^0 = \frac{s}{\sqrt{s+1}}V^0 + V_x^0 = 0.$$

inverse Laplace transform:

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st}}{\sqrt{s+1}} ds = \frac{1}{2\pi\sqrt{t}} e^{-t} \int_{-\infty}^{\infty} \frac{e^{iz}}{\sqrt{iz}} dz = \frac{e^{-t}}{\sqrt{\pi t}}$$

Dirichlet-Neumann relation:

$$v_x(0, t) = \frac{d}{dt} \left[\frac{e^{-t}}{\sqrt{\pi t}} \star v(0, t) \right] = \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}} v(0, \tau) d\tau.$$

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Navier-Stokes equations

$$\begin{cases} \rho_t + v_x = 0, \\ v_t + \rho_x = v_{xx}. \end{cases}$$

Zero initial value $(\rho, v)(x, 0) = 0, x > 0$.

Dirichlet-Neumann relation:

$$v_x(0, t) = \frac{d}{dt} \left[\frac{e^{-t}}{\sqrt{\pi t}} * v(0, t) \right] = \int_0^t \frac{e^{-(t-\tau)}}{\sqrt{\pi(t-\tau)}} v(0, \tau) d\tau.$$

Recover $\rho(0, t)$ from the continuity equation $\rho_t + v_x = 0$:

$$\rho(0, t) = \int_0^t v_x(0, s) ds = \int_0^t \int_0^s \frac{e^{-(s-\tau)}}{\sqrt{\pi(s-\tau)}} v(0, \tau) d\tau ds..$$

Therefore, only **one** boundary condition $v(0, t)$ for **well-posedness**.

Boundary relation for some dissipative systems

Remark: Convective Navier-Stokes equations

For the situation of evaporation $\Gamma > 0$, or condensation $\Gamma < 0$, the well-posed boundary conditions are different from each other and also from the above case of solid boundary:

$$\begin{cases} \rho_t + \Gamma \rho_x + v_x = 0, \\ v_t + \Gamma v_x + \rho_x = v_{xx}. \end{cases}$$

The study of the boundary relations naturally yields the well-posed boundary conditions.

Boundary relation for some dissipative systems

Wave equations

$$u_{tt} = u_{xx} + u_{yy}$$

can be written as a system of two first order equations

$$\begin{cases} u_t + u_x + v_y = 0, \\ v_t - v_x + u_y = 0, \end{cases} \quad v \equiv \int^y -(u_t + u_x) dy.$$

Dissipative wave equations

$$\begin{cases} u_t + u_x + v_y = \Delta u \\ v_t - v_x + u_y = \Delta v \end{cases}$$

Boundary relation for some dissipative systems

Dissipative wave equations

Laplace (t)-Fourier (y) (U, V)(x, ξ, s); Laplace (x)

(\hat{U}, \hat{V})(η, ξ, s):

$$\begin{pmatrix} s + \xi - \xi^2 + \eta^2 & i\eta \\ i\eta & s - \xi - \xi^2 + \eta^2 \end{pmatrix} \begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \begin{pmatrix} (1 - \xi)U^0 - U_x^0 \\ (-1 - \xi)V^0 - V_x^0 \end{pmatrix},$$

$$\begin{pmatrix} \hat{U} \\ \hat{V} \end{pmatrix} = \frac{1}{p(\xi, \eta, s)} \begin{pmatrix} s - \xi - \xi^2 + \eta^2 & -i\eta \\ -i\eta & s + \xi - \xi^2 + \eta^2 \end{pmatrix} \begin{pmatrix} (1 - \xi)U^0 - U_x^0 \\ (-1 - \xi)V^0 - V_x^0 \end{pmatrix}$$

Characteristic polynomial $p(\xi, \eta, s)$, in ξ variable,

$$\begin{aligned} p(\xi, \eta, s) &= \det \begin{pmatrix} s + \xi - \xi^2 + \eta^2 & i\eta \\ i\eta & s - \xi - \xi^2 + \eta^2 \end{pmatrix} \\ &= s^2 + \eta^2 + 2s\eta^2 + \eta^4 - (1 + 2s + 2\eta^2)\xi^2 + \xi^4 \end{aligned}$$

Boundary relation for some dissipative systems

Dissipative wave equations

$$p(\xi, \eta, s) = s^2 + \eta^2 + 2s\eta^2 + \eta^4 - (1 + 2s + 2\eta^2)\xi^2 + \xi^4$$

has four roots, λ_1, λ_2 have positive real roots and correspond to unstable modes:

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \equiv \left\{ \sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1 + 4s + \eta^2}}, \sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1 + 4s + \eta^2}}, \right. \\ \left. -\sqrt{\frac{1}{2} + s - \frac{1}{2}\sqrt{1 + 4s + \eta^2}}, -\sqrt{\frac{1}{2} + s + \frac{1}{2}\sqrt{1 + 4s + \eta^2}} \right\}. \quad (1)$$

Boundary relation for some dissipative systems

Dissipative wave equations

Dirichlet-Neumann relation in the transformed variables

$$U_x^0 = \frac{1}{2}(1 - \alpha - \beta)U^0 + \frac{i(1 + \alpha - \beta)(1 - 2\alpha + \alpha^2 - \beta^2)}{4\eta}V^0,$$

$$V_x^0 = \frac{1}{8(1 + 4s)\eta^3}i(\alpha - \beta)\left(-\alpha + \alpha^2 - \beta(1 + \beta)\right) \\ \left(1 + 4s + \beta^2 - 2\beta^3 + \alpha^2(-1 + 2\beta)\right) \\ \cdot [-1 + \beta^2 - 2s(2 + \alpha + \beta) + \alpha^2(1 + 2\beta) - 2\eta^2 - 2\beta\eta^2 \\ + 2\alpha(\beta + \beta^2 - \eta^2)]U^0 + \frac{1}{2}(-1 - \alpha - \beta)V^0,$$

Boundary relation for some dissipative systems

Dissipative wave equations The Dirichlet-Neumann relations in the transformed variables contain expressions which are polynomials in α and β where

$$\begin{cases} \alpha = \lambda_2 \equiv \sqrt{\left(\frac{1}{2} - \sqrt{s + 1/4}\right)^2 + \eta^2}, \\ \beta = \lambda_1 \equiv \sqrt{\left(\frac{1}{2} + \sqrt{s + 1/4}\right)^2 + \eta^2}. \end{cases} \quad (2)$$

As the functions α and β do not decay to zero for $s \in i\mathbb{R}$ as $|s| \rightarrow \infty$, instead we will invert the transforms for

$$\frac{\partial \alpha}{\partial s} = \left[\sqrt{\frac{1}{4} + s} - \frac{1}{2} \right] / \left[2\sqrt{\frac{1}{4} + s} \sqrt{\left(\frac{1}{2} - \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2} \right],$$

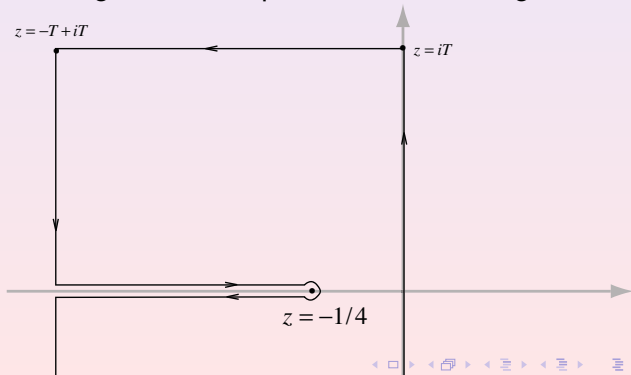
$$\frac{\partial \beta}{\partial s} = \left[\sqrt{\frac{1}{4} + s} + \frac{1}{2} \right] / \left[2\sqrt{\frac{1}{4} + s} \sqrt{\left(\frac{1}{2} + \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2} \right].$$

Boundary relation for some dissipative systems

Dissipative wave equations The Bromwich's integral of β_s for the inverse Laplace transform yielding exponential decay due to spectral gap, the real part of β_s is less than $-1/4$:

$$\frac{1}{2\pi i} \int_{\text{Re}(s)=0} \left(\frac{\partial \beta}{\partial s} \right) e^{st} ds = O(1) \frac{e^{-t/4}}{\sqrt{t}(1+|\eta|)}. \quad (3)$$

using the contour integral over the path illustrated in Figure 1:



Boundary relation for some dissipative systems

Dissipative wave equations α_s , which can be arbitrarily close to zero as $|\eta|$ varies. An entirely different, more sophisticated thinking is needed for computing the Bromwich integral,

$$\frac{1}{2\pi i} \int_{\operatorname{Re}(s)=0} \frac{\partial \alpha}{\partial s} e^{st} ds = \frac{1}{2\pi i} \int_{\operatorname{Re}(s)=0} \frac{1 - \frac{1}{2\sqrt{\frac{1}{4}+s}}}{2\sqrt{\frac{1}{2}+s} - \sqrt{\frac{1}{4}+s} + \eta^2} e^{st} ds. \quad (4)$$

The function α_s is analytic in the cut domain

$$\mathbb{C} \setminus \left(\{z = -x^2 \pm ix : x \geq |\eta|\} \cup (-\infty, -1/4] \right).$$

To compute the Bromwich integral we integrate α_s along the boundary of this domain, the contours in Figure 2:

Boundary relation for some dissipative systems

Dissipative wave equations

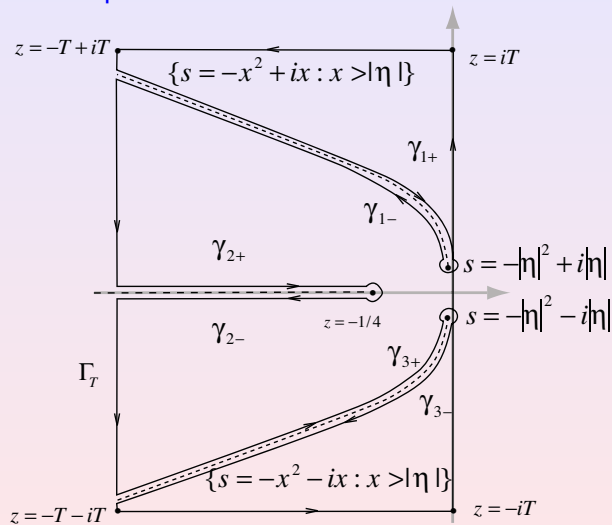


Figure 2. The contour of the path integral of Γ_T .

Boundary relation for some dissipative systems

Dissipative wave equations

The integral along $\gamma_{2\pm}$ decays exponentially in t due to the spectral gap.

The integrals along $\gamma_{1\pm}$ is **characteristic** in the sense that α_s can be arbitrarily close to the origin as η varies. We set

$$\sqrt{1/2 + s - \sqrt{1/4 + s + \eta^2}} = \pm i\tau$$

so that $-\infty < \tau < \infty$ along $\gamma_{1\pm}$.

Similarly, along $\gamma_{3\pm}$, set

$$\sqrt{1/2 + s - \sqrt{1/4 + s + \eta^2}} = i\tau.$$

We will integrate in τ instead of s . By long computations, we have

Boundary relation for some dissipative systems

Dissipative wave equations

$$\int_{\gamma_{1+} + \gamma_{1-}} \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + s}}{2\sqrt{\frac{1}{4} + s} \sqrt{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2}} e^{st} ds$$
$$= 2i \int_0^\infty e^{-t(\tau^2 + \eta^2)} \left(\cos\left(t\sqrt{\tau^2 + \eta^2}\right) + i \sin\left(t\sqrt{\tau^2 + \eta^2}\right) \right) d\tau;$$

$$\int_{\gamma_{3+} + \gamma_{3-}} \frac{-\frac{1}{2} + \sqrt{\frac{1}{4} + s}}{2\sqrt{\frac{1}{4} + s} \sqrt{\left(-\frac{1}{2} + \sqrt{\frac{1}{4} + s}\right)^2 + \eta^2}} e^{st} ds$$
$$= -2 \int_0^\infty e^{-t(\eta^2 + \tau^2)} \left(i \cos\left(t\sqrt{\eta^2 + \tau^2}\right) + \sin\left(t\sqrt{\eta^2 + \tau^2}\right) \right) d\tau.$$

Boundary relation for some dissipative systems

Dissipative wave equations

The integral along $\gamma_{1\pm}$ and $\gamma_{3\pm}$ is inverted with respect to the Fourier variable η to yield the expression:

$$\mathbf{W}_2(y, t) \equiv -\frac{1}{\pi^2} \int_{\mathbb{R}^2} e^{iy\eta} \int_0^\infty e^{-t(\eta^2 + \tau^2)} \cos\left(t\sqrt{\eta^2 + \tau^2}\right) d\tau d\eta.$$

This function can be identified with the convolution of the 2-dimensional heat kernel with the solution u of the following initial value problem of a wave equation at time t :

$$\begin{cases} \left(\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2} \right) u(y', z', t') = 0, \\ u(y', z', 0) = -\frac{1}{\pi^2} \delta(y') \delta(z'), \\ u_{t'}(y', z', 0) = 0. \end{cases}$$

Boundary relation for some dissipative systems

Dissipative wave equations

The Green's function u for the 2-dimensional wave equation has been constructed by the Hadamard's descending method in terms of the Kirchhoff formula for 3-dimensional wave equation. The convolution with the heat kernel becomes:

$$\begin{aligned} & -\frac{1}{\pi^2} \int_{\mathbb{R}} e^{iy\eta} \int_0^\infty e^{-t(\eta^2 + \tau^2)} \cos\left(t\sqrt{\eta^2 + \tau^2}\right) d\tau d\eta \\ & = -\frac{1}{\pi^2} \partial'_t \left[\frac{1}{4\pi t'} \iint_{|\vec{y}-\vec{z}|=t'} k(\vec{z}, t) dS_{\vec{y}} \right] \Big|_{t'=t}, \end{aligned}$$

where

$$\begin{cases} \vec{y} = (y, 0, 0) \in \mathbb{R}^3, \\ \vec{z} = (z^1, z^2, z^3) \in \mathbb{R}^3, \\ k(\vec{z}, t) \equiv \frac{e^{-\frac{|z^1|^2 + |z^2|^2}{4t}}}{4\pi t}. \end{cases}$$

Boundary relation for some dissipative systems

Dissipative wave equations

Conclusion

The boundary relation for the dissipative wave equation is expressed in terms of convolution of local (exponentially decaying terms) operator with Hadamard-type operator.