

Long-time asymptotics for the Camassa–Holm equation

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The Camassa-Holm equation

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

(another form: $m_t + (m \frac{\partial}{\partial x} + \frac{\partial}{\partial x} m)u = 0$ with $m = u - u_{xx} + \omega$)

- abstract bi-Hamiltonian equation (Fokas, Fuchssteiner, 1981)
 - unidirectional propagation of 2-dim waves in shallow water (Camassa, Holm, 1993); constant ω is related to the critical wave speed $\sqrt{gh_0}$
 - g is the gravity acceleration,
 - h_0 is the undisturbed water depth.
 - waves in hyperelastic rods (Dai, 1998)
 - geodesic flow on diffeomorphism group of the line (Misiolek, 1998)
- ▷ $\omega = 0$: **peakons** $c \cdot e^{-|x-ct|}$
- ▷ $\omega > 0$: **smooth solitary waves**

Shallow water wave models

- result from approximations to “full” equations (Euler, Green–Naghdi) governing the motion of inviscid fluid whose surface can exhibit gravity wave propagation

- small parameters: $\varepsilon = \frac{a}{h}$, $\mu = \frac{h^2}{\lambda^2}$, where

- a is the typical amplitude
- h is the mean depth
- λ is the typical wavelength

- ▷ shallow water scaling: $\mu \ll 1$ **weakly dispersive**
- ▷ long-wave: $\varepsilon \ll 1$ **weakly nonlinear**

Shallow water wave models

- long-wave regime: $\mu \ll 1$, $\varepsilon = O(\mu)$; **balance between nonlinearity and dispersion**

- ▷ Korteweg-de Vries (KdV) eq.:

$$u_t + u_x + \varepsilon \frac{3}{2} uu_x + \mu \frac{1}{6} u_{xxx} = 0$$

- ▷ family of Benjamin, Bona, Mahoney (BBM) eqs. (same accuracy as KdV):

$$u_t + u_x + \varepsilon \frac{3}{2} uu_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = 0$$

- “Camassa–Holm scaling”: $\mu \ll 1$, $\varepsilon = O(\sqrt{\mu})$.
 - **more nonlinear than dispersive**; could allow **breaking waves**
 - generalization of the BBM equations (approximation of the same order $O(\mu^2)$ as the BBM)

$$u_t + u_x + \varepsilon \frac{3}{2} uu_x + \mu(\alpha u_{xxx} + \beta u_{xxt}) = \varepsilon \mu(\gamma uu_{xxx} + \delta u_x u_{xx})$$

(some conditions on $\alpha, \beta, \gamma, \delta$)

Integrable models (bi-Hamiltonian structure, Lax pair)

- in the BBM family, only KdV is integrable
- in the “Camassa–Holm scaling”: two equations are integrable. After rescaling:
 - ▷ the Camassa–Holm (CH) equation (1993):

$$u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \quad \omega \in \mathbb{R}$$

- ▷ the Degasperis–Procesi (DP) equation (1999):

$$u_t - u_{txx} + 3\omega u_x + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \quad \omega \in \mathbb{R}$$

- ★ in the family of equations (*b*-family)

$$u_t - u_{txx} + u_x + (b + 1)uu_x = bu_x u_{xx} + uu_{xxx},$$

only integrable are the Camassa–Holm (**b=2**) and the Degasperis–Procesi (**b=3**) equations.

- hydrodynamical relevance of CH and DP equations: Johnson (2002), Dullin, Gottwald, Holm (2003), Constantin and Lannes (2009).
- **inconsistency of CH with shallow water theory**: Blatt, Mikhailov (arXiv, 2010)

CH ($\omega = 1$) on $(-\infty, \infty)$ with fast decaying initial data

Let $u(x, t)$ be the solution of the initial-value problem for the Camassa-Holm equation (for $\omega = 1$):

- $u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
- $u(x, 0) = u_0(x)$

Let $m(x, t) := u(x, t) - u_{xx}(x, t)$.

Assumptions:

- $u_0(x) \rightarrow 0$ as $|x| \rightarrow \infty$
- $m(x, 0) + 1 > 0$ for all $x \in \mathbb{R}$ (then $m(x, t) + 1 > 0$ for all x, t)

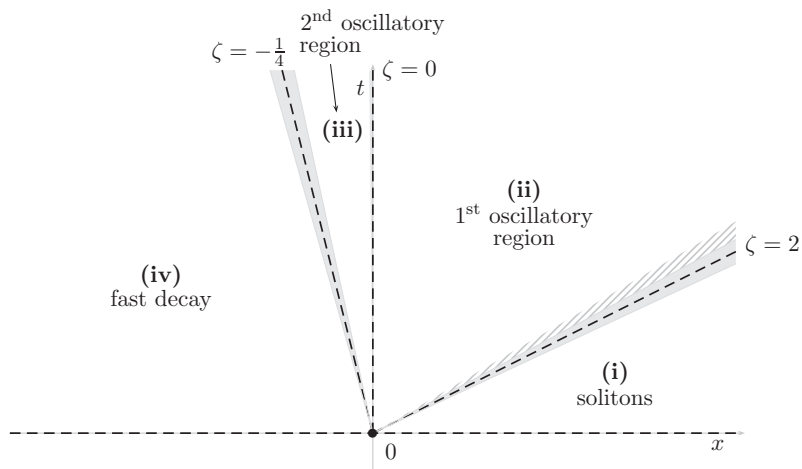
Question

How does $u(x, t)$ behave for large t ?

Answer

- **four sectors** in the (x, t) half-plane where $u(x, t)$ behaves differently for large t , depending on the magnitude of $\zeta = \frac{x}{t}$
- **transition zones** (2 Painlevé zones; collisionless shock region)

Long-time asymptotics



Four sectors and transition zones in the (x, t) -half-plane, $\zeta = \frac{x}{t}$.
Painlevé zones: (a) $|\zeta - 2|t^{2/3} < C$; (b) $|\zeta + \frac{1}{4}|t^{2/3} < C$

Long-time asymptotics

Sector (i): $u(x, t)$ looks like a **finite train of solitons**

Sector (ii): $u(x, t)$ looks like a **slowly decaying modulated oscillation**

Sector (iii): $u(x, t) \sim$ the sum of **two decaying modulated oscillations**

Sector (iv): $u(x, t)$ is **fast decaying**

- transitions: in terms of solutions of **Painlevé II** equation

$$v''(s) = sv(s) + 2v^3(s)$$

Lax pair:

$$\begin{aligned}\Psi_{xx} &= \frac{1}{4}\Psi + \lambda(m+1)\Psi, \\ \Psi_t &= \left(\frac{1}{2\lambda} - u\right)\Psi_x + \frac{1}{2}u_x\Psi\end{aligned}$$

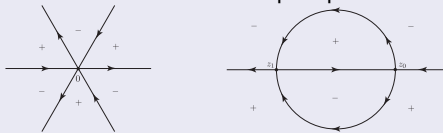
Inverse scattering transform method **in the RHP form**

- using
 - ▷ the **Lax pair** associated to the CH equation
- construct
 - ▷ a multiplicative matrix **Riemann-Hilbert** problem (RHP)
- obtain
 - ▷ a representation of the solution $u(x, t)$ of the CH equation in terms of the solution $\mu(x, t; k)$ of the associated RHP
- obtain
 - ▷ the long-time asymptotics of $u(x, t)$ via the Deift-Zhou **nonlinear steepest descent** method.

Riemann–Hilbert problem

Data

- An oriented **contour** Σ in the complex plane.



- A **jump** along Σ , i.e., a function $J: \Sigma \rightarrow GL(2, \mathbb{C})$
- **Points** $\kappa_1, \dots, \kappa_n \notin \Sigma$ with prescribed **polar parts**.

Problem

Find a matrix-valued function $M(k)$ such that

- M is meromorphic on $\mathbb{C} \setminus \Sigma$. (analyticity condition)
- $M_-(k) = M_+(k)J(k)$ for $k \in \Sigma$. (jump condition)
- M satisfies residue conditions at $\kappa_1, \dots, \kappa_n$. (residue conditions)
- $M(\infty) = I$. (normalization condition)

▷ $J = J(x, t; k) \mapsto M = M(x, t; k)$

▷ **analogue of contour integrals**: Bessel, Airy, Hermite, . . .

Soliton sector: $x/t > 2$

$u(x, t)$ behaves like the sum of N solitons:

- ▷ parameters: $\{\nu_j, \gamma_j\}_{j=1}^N$ (**discrete spectrum of x -eqn.** from Lax pair)

$$\blacksquare u(x, t) = \sum_{j=1}^N u_j(x - c_j t - x_{j0}) + O\left(\frac{1}{t^l}\right), \quad c_j = \frac{2}{1 - 4\nu_j^2}$$

where $u_j(\cdot)$: 1-soliton ($\nu = \nu_j$) in **parametric form (cf. KdV)**

- ▷ $u(X) = U(Y(X))$
- ▷ $U(Y) = \frac{16\nu^2}{1 - 4\nu^2} \frac{1}{1 + 4\nu^2 + (1 - 4\nu^2) \cosh Y}$,
- ▷ $X(Y) = Y + \log \frac{1 - 2\nu + (1 + 2\nu)e^{-2\nu Y}}{1 + 2\nu + (1 - 2\nu)e^{-2\nu Y}}$
- ▷ $x_{j0} = \log \left[\frac{\gamma_j}{2\nu_j} \frac{1+2\nu_j}{1-2\nu_j} \prod_{l=j+1}^N \left(\frac{\nu_j - \nu_l}{\nu_j + \nu_l} \right)^2 \prod_{l=j+1}^N \left(\frac{1+2\nu_l}{1-2\nu_l} \right)^2 \right]$

Here $0 < \nu_j < 1/2$ and thus the soliton velocity c_j : $c_j > 2$.

Decaying modulated oscillations ($\zeta = x/t$)

(ii) $0 \leq \zeta < 2$:

$$\blacksquare u(x, t) = \frac{c_1^{(0)}}{\sqrt{t}} \times \sin \left(c_2^{(0)} t + c_3^{(0)} \log t + c_4^{(0)} \right) + o\left(\frac{1}{\sqrt{t}}\right),$$

$$\triangleright c_l^{(0)} = c_l^{(0)}(\kappa_0(\zeta); \text{scatt.data})$$

$$\triangleright \kappa_0^2(\zeta) = \frac{\sqrt{1+4\zeta}-1-\zeta}{4\zeta}$$

$$\triangleright u_0(x) \mapsto \{r(k); \{v_j, \gamma_j\}\}$$

(iii) $-\frac{1}{4} < \zeta < 0$: sum of 2 oscillations

$$\blacksquare u(x, t) = \sum_{j=0,1} \frac{c_1^{(j)}}{\sqrt{t}} \times \sin \left(c_2^{(j)} t + c_3^{(j)} \log t + c_4^{(j)} \right) + o\left(\frac{1}{\sqrt{t}}\right),$$

$$\triangleright c_l^{(j)} = c_l^{(j)}(\kappa_0(\zeta), \kappa_1(\zeta); \text{scatt.data})$$

$$\triangleright \kappa_1^2(\zeta) = -\frac{\sqrt{1+4\zeta}+1+\zeta}{4\zeta}$$

(iv) $\zeta < -1/4$: fast decay

$$\blacksquare u = o\left(\frac{1}{(|x|+t)^n}\right)$$

1st transition zone

close to $x/t = 2$

For $|\frac{x}{t} - 2|t^{2/3} < C, C > 0,$

$$\blacksquare u(x, t) \sim -\left(\frac{4}{3}\right)^{2/3} \frac{1}{t^{2/3}} (u^2(s) - u'(s)),$$

where

$$\triangleright s = 6^{-1/3} \left(\frac{x}{t} - 2\right) t^{2/3},$$

$\triangleright u(s)$ is the solution of the Painlevé II equation

$$u''(s) = su(s) + 2u^3(s)$$

fixed by its asymptotics as $s \rightarrow +\infty$:

$$u(s) \sim -r(0) \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{2}{3}s^{3/2}}$$

$\triangleright r(k)$ reflection coefficient of x -eqn. from Lax pair

2nd transition zone

close to $x/t = -1/4$

For $|\frac{x}{t} + \frac{1}{4}|t^{2/3} < C, C > 0$:

■ $u(x, t) \sim \frac{12^{1/6}}{t^{1/3}} u_1(s_1) \sin \psi(s_1, t),$

▷ $s_1 = -\left(\frac{16}{3}\right)^{1/3} \left(\frac{x}{t} + \frac{1}{4}\right) t^{2/3},$

▷ $\psi(s_1, t) = -\frac{3\sqrt{3}}{4} t - \frac{3^{5/6}}{24^{2/3}} s_1 t^{1/3} + \Delta,$

▷ $u_1(s)$ is the solution of the Painlevé II equation fixed by its asymptotics as $s \rightarrow +\infty$:

$$u_1(s) \sim \left| r\left(\frac{\sqrt{3}}{2}\right) \right| \frac{1}{2\sqrt{\pi}} s^{-\frac{1}{4}} e^{-\frac{2}{3}s^{3/2}}$$

▷ Δ : in terms of scattering data

Specific features of RHP for CH

- In order to control analytic properties of eigenfunctions on spectral parameter: **two versions** of the Lax pair
 - ▷ for large k (where $k^2 = -\frac{1}{4} - \lambda$)
 - ▷ for k near $\pm \frac{i}{2}$
- Solution $u(x, t)$ of the CH equation: from the evaluation of the solution of the RH problem for k near $\frac{i}{2}$
- **Dependence on u of the exponential factor** in the RH problem: requires introducing auxiliary scale, which leads to implicit (**parametric**) formulas for $u(x, t)$ (even for pure soliton solutions)

Eigenfunctions for large k

A 2×2 variant of the Lax pair: $(k^2 = -\frac{1}{4} - \lambda, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix})$

$$\Phi_x + ikp_x \sigma_3 \Phi = U\Phi, \quad \Phi_t + ikp_t \sigma_3 \Phi = V\Phi$$

$$U(x, t; k) = U_0(x, t) + \frac{1}{k}U_{-1}(x, t),$$

$$V(x, t; k) = V_0(x, t) + \frac{1}{k}V_{-1}(x, t) + \frac{k}{4k^2+1}V_1(x, t)$$

are 2×2 matrices dependent on $u(x, t)$.

Eigenfunctions: $\hat{\Phi} = \Phi e^{ikp(x,t,k)\sigma_3}$ $(e^{\hat{\sigma}_3 A} \equiv e^{\sigma_3 A e^{-\sigma_3}})$

$$\hat{\Phi}_{\pm}(x, t, k) = I + \int_{\pm\infty}^x e^{-ik(p(x,t,k)-p(y,t,k))\hat{\sigma}_3} (U\hat{\Phi}_{\pm}) dy$$

$\hat{\Phi} \rightarrow I$ as $k \rightarrow \infty$.

Here

$$p(x, t, k) := x - \int_x^{\infty} \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi - \frac{2}{1 + 4k^2} t$$

$p(x, t, k)$: controls large- k behavior of Φ ; depends on u

Eigenfunctions for k near $\pm i/2$ (λ near 0)

Lax pair:

$$\Phi_x^0 + ik\sigma_3\Phi^0 = U^0\Phi^0, \quad \Phi_t^0 - \frac{4ik}{1+k^2}\sigma_3\Phi^0 = V^0\Phi^0$$

$$U^0(x, t; k) = -\frac{1+4k^2}{8ik}m(x, t) \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad U^0(x, t; k)|_{k=\pm i/2} = 0.$$

Eigenfunctions: $\Phi^0 = \hat{\Phi}^0 e^{-ik(x - \frac{2t}{1+4k^2})\sigma_3}$

$$\hat{\Phi}_{\pm}^0(x, t, k) = I + \int_{\pm\infty}^x e^{-ik(x-y)\hat{\sigma}_3} (U^0 \hat{\Phi}_{\pm}^0) dy$$

$\hat{\Phi}_{\pm}^0(x, t, \pm i/2) = I$ for all x, t .

In terms of $\hat{\Phi}$: specific matrix structure at $k = i/2$

$$\begin{pmatrix} \hat{\Phi}_{-}^{(1)} & \hat{\Phi}_{+}^{(2)} \end{pmatrix} \left(x, t, \frac{i}{2}\right) = F(x, t) \begin{pmatrix} e^{-\frac{1}{2} \int_{-\infty}^x \sqrt{m+1}-1} & 0 \\ 0 & e^{-\frac{1}{2} \int_x^{\infty} \sqrt{m+1}-1} \end{pmatrix},$$

where $F = \frac{1}{2} \begin{pmatrix} g + 1/g & g - 1/g \\ g - 1/g & g + 1/g \end{pmatrix}$, $g = (m + 1)^{1/4}$

- Scattering matrix

$$\hat{\Phi}_+ = \hat{\Phi}_- e^{-ikp(x,t,k) \hat{\sigma}_3} \begin{pmatrix} \bar{a}(k) & b(k) \\ \bar{b}(k) & a(k) \end{pmatrix}, \quad k \in \mathbb{R}$$

- ▷ reflection coefficient: $r(k) = \frac{b(k)}{\bar{a}(k)}$

- Discrete spectrum: $\{k_j\}_{j=1}^N : a(k_j) = 0$:

- ▷ simple; $N < \infty$

- ▷ $k_j = i\nu_j$, $0 < \nu_j < \frac{1}{2}$

- ▷ $\hat{\Phi}_-^{(1)}(x, t, i\nu_j) = \chi_j e^{2\nu_j p(x, t, i\nu_j)} \hat{\Phi}_+^{(2)}(x, t, i\nu_j)$

$u_0(x)$ determines $a(k), b(k)$

Riemann-Hilbert problem in (x, t)

$$M(x, t, k) := \begin{cases} \begin{pmatrix} \frac{\hat{\Phi}_-^{(1)}}{a(k)} & \hat{\Phi}_+^{(2)} \\ \hat{\Phi}_+^{(1)} & \frac{\hat{\Phi}_-^{(2)}}{\bar{a}(k)} \end{pmatrix}, & \text{Im } k > 0 \\ \begin{pmatrix} \hat{\Phi}_-^{(1)} & \hat{\Phi}_+^{(2)} \\ \frac{\hat{\Phi}_-^{(1)}}{a(k)} & \frac{\hat{\Phi}_-^{(2)}}{\bar{a}(k)} \end{pmatrix}, & \text{Im } k < 0 \end{cases}$$

- $M_- = M_+ e^{-ikp(x,t,k)\hat{\sigma}_3} \begin{pmatrix} 1 & -r(k) \\ \bar{r}(k) & 1 - |r(k)|^2 \end{pmatrix}, \quad k \in \mathbb{R}$

- $M \rightarrow I$ as $k \rightarrow \infty$

- $\text{Res}_{k=i\nu} M^{(1)} = i\gamma_j e^{-2\nu_j p(x,t,i\nu_j)} M^{(2)}(x, t, i\nu_j)$

- $M \sim \frac{\alpha(x,t)}{ik} \begin{pmatrix} -c & -1 \\ c & 1 \end{pmatrix}, k \rightarrow 0$

- structure at $k = \frac{i}{2}$: $M(x, t, \frac{i}{2}) = \begin{pmatrix} \xi & \eta \\ \eta & \xi \end{pmatrix} \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix}$

$$p(x, t, k) = x - \int_x^\infty \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi - \frac{2}{1 + 4k^2 t}$$

Riemann-Hilbert problem in (y, t)

Introduce **new scale**: $y(x, t) := x - \int_x^\infty \left(\sqrt{m(\xi, t) + 1} - 1 \right) d\xi$.

Then $p = p(y, t) = y - \frac{2t}{1 + 4k^2}$.

RH problem $(y, t; 2 \times 1)$: Given $r(k), k \in \mathbb{R}; \{\nu_j, \gamma_j\}_{j=1}^N$
($0 < \nu_j < \frac{1}{2}, \gamma_j > 0$), find $\mu(y, t, k) = (\mu_1 \ \mu_2)$ s.t.

- $\mu_-(y, t, k) = \mu_+(y, t, k)J(y, t, k), k \in \mathbb{R}$,
where

$$J = \begin{pmatrix} 1 & -r(k)e^{-2ik\left(y - \frac{2t}{1+4k^2}\right)} \\ \bar{r}(k)e^{2ik\left(y - \frac{2t}{1+4k^2}\right)} & 1 - |r|^2 \end{pmatrix}$$

- $\mu \rightarrow (1 \ 1), k \rightarrow \infty$
- $\text{Res}_{k=i\nu} \mu_1(y, t, k) = i\gamma_j e^{-2\nu_j\left(y - \frac{2t}{1-4\nu_j^2}\right)} \mu_2(y, t, i\nu_j)$
- $\mu(-k) = \mu(k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Obtaining $u(x, t)$ from solution of RHP

Evaluate $\mu = (\mu_1 \ \mu_2)(y, t, k)$ for k near $\frac{i}{2}$. Then $u(x, t)$ can be obtained **in a parametric form**:

- $$u(y, t) = \frac{1}{2i} \lim_{k \rightarrow \frac{i}{2}} \left(\frac{\mu_1(y, t; k) \mu_2(y, t; k)}{\mu_1(y, t; \frac{i}{2}) \mu_2(y, t; \frac{i}{2})} - 1 \right) \frac{1}{k - \frac{i}{2}}$$

- $$x(y, t) = y + \log \frac{\mu_1(y, t, i/2)}{\mu_2(y, t, i/2)}$$

Nonlinear steepest descent method

Nonlinear steepest descent method (Deift / Zhou) is applicable to RHP whose jump matrix depends on a large parameter t and **oscillates** with t . Aim: asymptotics of the solution as $t \rightarrow \infty$.

Nonlinear steepest descent method for oscillatory RHP

- *Deformations* of contour Σ , *approximation* of jump matrix $J(y, t; k)$ in order to arrive at a new RHP with jump matrix J^* approaching I as $t \rightarrow +\infty$ on the most part of the contour (outside **stationary phase points**)
- *Scaling* **near stationary phase points** in order to arrive at a **model** RHP, RH^{mod} , with piecewise constant (in k) jump matrix: $J^* \sim J^{\text{mod}}$ as $t \rightarrow +\infty$
- Construct **explicit** solution of model problem (in terms of special functions)

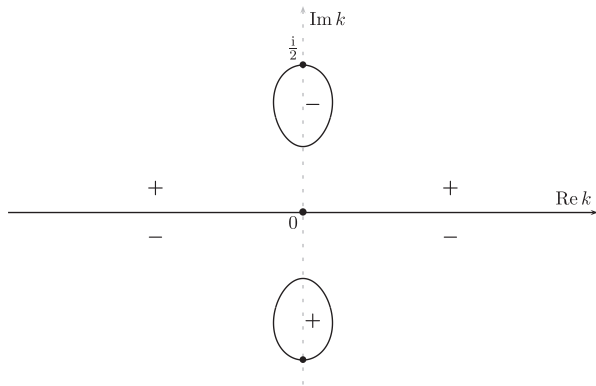
Asymptotics: signature tables (1)

Exponential factor in the jump matrix for RH problem dictate the contour deformations:

$$e^{2ik\left(y - \frac{2t}{1+4k^2}\right)} \equiv e^{2it\theta}, \quad \theta(y, t, k) = \hat{\zeta}k - \frac{2k}{1+4k^2}. \quad \text{Here } \hat{\zeta} = \frac{y}{t}.$$

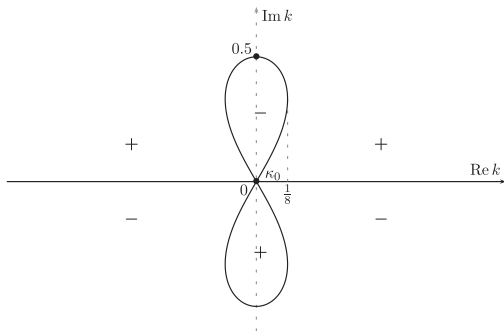
Sign of $\text{Im } \theta$:

(i) $\hat{\zeta} > 2$



Asymptotics: signature tables (2)

$$\hat{\zeta} = 2$$

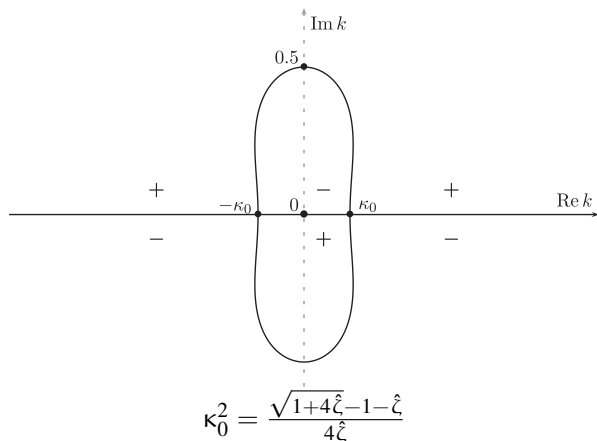


$$\kappa_0^2 = \frac{\sqrt{1+4\hat{\zeta}}-1-\hat{\zeta}}{4\hat{\zeta}}$$

• 1st Painlevé zone: $\hat{\zeta} \rightarrow 2 \implies \kappa_0 \rightarrow 0$

Asymptotics: signature tables (3)

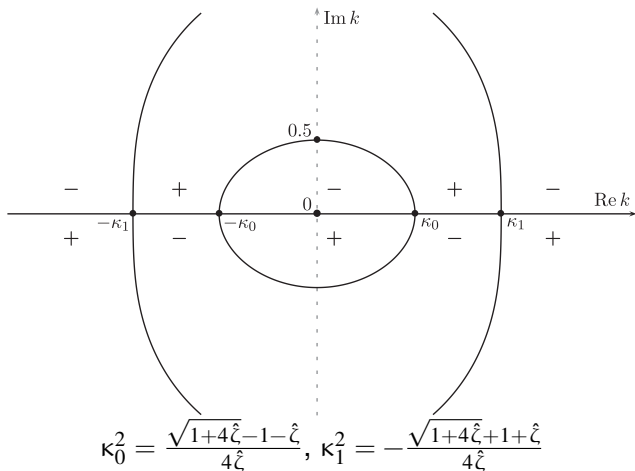
(ii) oscillatory region: $0 \leq \hat{\zeta} < 2$. Here $0 < \kappa_0 < \frac{1}{2}$.



Needs: scalar RHP $\delta_+ = \delta_- (1 - |r|^2)$, $-\kappa_0 < k < \kappa_0$

Asymptotics: signature tables (4)

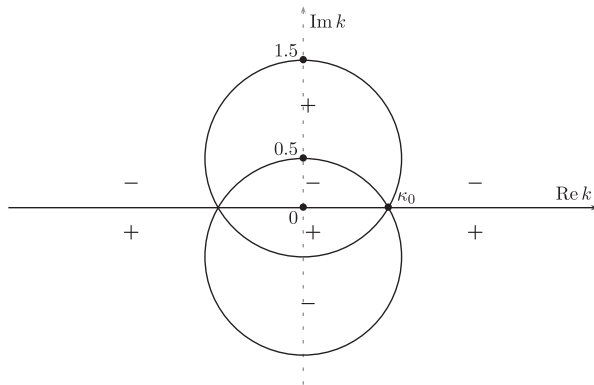
(iii) oscillatory region: $-\frac{1}{4} < \hat{\zeta} < 0$



Here $\frac{1}{2} < \kappa_0 < \frac{\sqrt{3}}{2} < \kappa_1 < \infty$.

Asymptotics: signature tables (5)

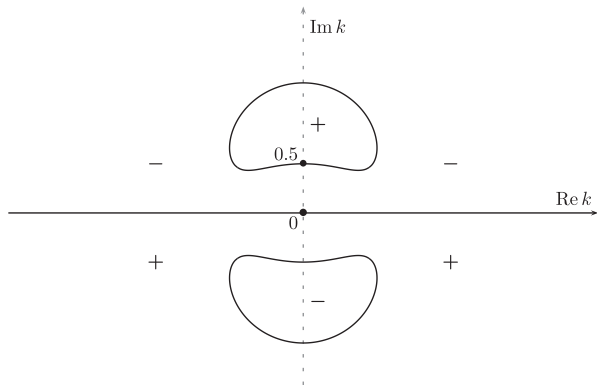
$$\hat{\zeta} = -\frac{1}{4}$$



- **2nd Painlevé zone:** $\hat{\zeta} \rightarrow -\frac{1}{4} \implies \kappa_j \rightarrow \frac{\sqrt{3}}{2}, j = 0, 1$

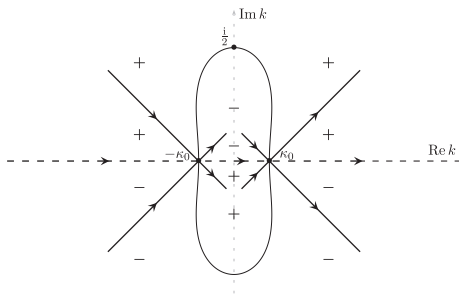
Asymptotics: signature tables (6)

(iv) $\hat{\zeta} < -\frac{1}{4}$



Here there are no real stationary points.

(ii) $0 < x/t < 2$: two crosses



Approximation of phase: $\theta(k) = \theta(\kappa_0) + \alpha(k - \kappa_0)^2 (1 + O(k - \kappa_0))$

Scaling: $k - \kappa_0 = \frac{\hat{k}}{\sqrt{4\alpha t}}$. Here $\theta(\kappa_0) = -\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2}$, $\alpha = \frac{8\kappa_0(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3}$

Approximation of jump matrix: $\delta(k)e^{-i\theta} \sim \delta_{\kappa_0} \hat{k}^{ih_0} e^{-\frac{ik^2}{4}}$, where

$$\triangleright h_0 = -\frac{1}{2\pi} \log(1 - |r(\kappa_0)|^2), \quad \chi(k) = \frac{1}{2\pi i} \int_{-\kappa_0}^{\kappa_0} \log\left(\frac{1 - |r(s)|^2}{1 - |r(\kappa_0)|^2}\right) \frac{ds}{s-k}$$

$$\triangleright \delta_{\kappa_0} = \left(\frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} t\right)^{-\frac{ih_0}{2}} e^{i\frac{16\kappa_0^3}{(1+4\kappa_0^2)^2} t} e^{\chi(\kappa_0)}$$

Asymptotics in the oscillatory sector $0 < x/t < 2$

$$\kappa_0 = \kappa_0(\zeta) = \frac{\sqrt{1+4\zeta} - 1 - \zeta}{4\zeta}$$

$$\blacksquare u(x, t) = \frac{c_1^{(0)}}{\sqrt{t}} \cdot \sin \left(c_2^{(0)} t + c_3^{(0)} \log t + c_4^{(0)} \right) + o\left(\frac{1}{\sqrt{t}}\right),$$

where

$$\triangleright c_1^{(0)} = - \left(\frac{32h_0\kappa_0}{(3-4\kappa_0^2)(1+4\kappa_0^2)} \right)^{\frac{1}{2}}$$

$$\triangleright c_2^{(0)} = \frac{32\kappa_0^3}{(1+4\kappa_0^2)^2}$$

$$\triangleright c_3^{(0)} = -h_0$$

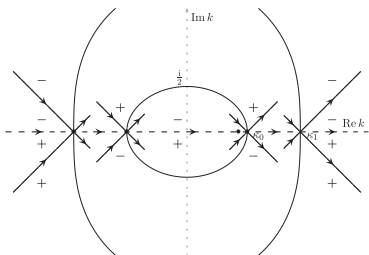
$$\triangleright c_4^{(0)} = \frac{\pi}{4} - \arg r(\kappa_0) + \arg \Gamma(ih_0) + \frac{1}{\pi} \int_{-\kappa_0}^{\kappa_0} \log |\kappa_0 - s| ds \log(1 - |r|^2) -$$

$$h_0 \log \frac{128\kappa_0^3(3-4\kappa_0^2)}{(1+4\kappa_0^2)^3} + \frac{4}{\pi} \kappa_0 \int_{-\kappa_0}^{\kappa_0} \frac{\log(1-|r(\kappa_0)|^2)}{1+4s^2} ds +$$

$$4 \sum_{j=1}^N \arctan \left(\frac{\nu_j}{\kappa_0} \right) + 4\kappa_0 \sum_{j=1}^N \log \frac{1+2\nu_j}{1-2\nu_j}$$

RHP deformations

(iii) $-1/4 < x/t < 0$: four crosses



$$\blacksquare u(x, t) = \sum_{j=0,1} \frac{c_1^{(j)}}{\sqrt{t}} \cdot \sin \left(c_2^{(j)} t + c_3^{(j)} \log t + c_4^{(j)} \right) + o\left(\frac{1}{\sqrt{t}}\right),$$

where

$$\triangleright c_1^{(j)} = - \left(\frac{32 h_j \kappa_j}{|3 - 4\kappa_j^2|(1 + 4\kappa_j^2)} \right)^{\frac{1}{2}} \quad \kappa_0(\zeta) = \frac{\sqrt{1 + 4\zeta} - 1 - \zeta}{4\zeta}$$

$$\triangleright c_2^{(j)} = \frac{32\kappa_j^3}{(1 + 4\kappa_j^2)^2} \quad \kappa_1(\zeta) = -\frac{\sqrt{1 + 4\zeta} + 1 + \zeta}{4\zeta}$$

$$\triangleright c_3^{(j)} = (-1)^j h_j$$

$$\triangleright c_4^{(j)} = c_4^{(j)}(\kappa_0, \kappa_1; r(k); \{v_j\}) = \dots$$

Scaling

$$\triangleright s = 6^{-1/3} \left(\frac{y}{t} - 2 \right) t^{2/3}, \quad \hat{k} = (6t)^{1/3} k$$

Then $e^{t\theta} \sim e^{\frac{4}{3}\hat{k}^3 + s\hat{k}}$ as in the RHP for the Painlevé II eq.

$$\blacksquare u(x, t) \sim -\left(\frac{4}{3}\right)^{2/3} \frac{1}{t^{2/3}} (v^2(s) - v'(s)),$$

$\triangleright v(s)$ solves Painlevé II and

$$v(s) \sim -r(0)Ai(s) \sim -r(0) \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{2}{3}s^{3/2}}$$

as $s \rightarrow \infty$

$$\triangleright s_1 = -\left(\frac{16}{3}\right)^{1/3} \left(\frac{y}{t} + \frac{1}{4}\right) t^{2/3}, \quad \hat{k} = -\left(\frac{3t}{16}\right)^{1/3} \left(k \mp \frac{\sqrt{3}}{2}\right)$$

$$\text{Then } t\theta \sim t\theta \left(\pm \frac{\sqrt{3}}{2}\right) + \frac{4}{3}\hat{k}^3 + s_1\hat{k}$$

$$\blacksquare u(x, t) \sim \frac{12^{1/6}}{t^{1/3}} v_1(s_1) \sin \psi(s_1, t),$$

$$\triangleright \psi(s_1, t) = -\frac{3\sqrt{3}}{4}t - \frac{3^{5/6}}{2^{4/3}}s_1 t^{1/3} + \Delta,$$

$$\triangleright v_1(s) \text{ solves PII and } v_1(s) \sim \left| r\left(\frac{\sqrt{3}}{2}\right) \right| \frac{1}{2\sqrt{\pi}} s^{-1/4} e^{-\frac{2}{3}s^{3/2}},$$

$$\triangleright \Delta = -\frac{4\sqrt{3}}{\pi} \int_0^\infty \frac{\log(1-|r(\xi)|^2)}{1+4\xi^2} d\xi + \arg r\left(\frac{\sqrt{3}}{2}\right) +$$

$$\frac{1}{\pi} \int_{-\infty}^\infty \frac{\log(1-|r(\xi)|^2)}{\xi - \frac{\sqrt{3}}{2}} d\xi - 4 \sum_{j=1}^N \arg\left(\frac{\sqrt{3}}{2} + i\kappa_j\right) - 2\sqrt{3} \sum_{j=1}^N \log\left(\frac{1+2\kappa_j}{1-2\kappa_j}\right)$$

CH on the half-line for fast decaying IB data

Let $u(x, t)$ be the solution of the initial-boundary-value problem for the Camassa-Holm equation (for $\omega = 1$):

- $u_t - u_{txx} + 2u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}$.
- $u(x, 0) = u_0(x)$.
- $u(0, t) = v_0(t)$, $u_x(0, t) = v_1(t)$, $u_{xx}(0, t) = v_2(t)$ **compatible**

▷ $u_0(x) \rightarrow 0$ as $x \rightarrow +\infty$, $v_j(t) \rightarrow 0$ as $t \rightarrow \infty$

There are

- **two sectors** in the (x, t) quarter-plane where $u(x, t)$ behaves differently for large t , depending on the magnitude of $\zeta = \frac{x}{t}$
- **transition zones**

The structure of the asymptotics is similar to the whole line case, but **parameters** are determined by initial and boundary conditions.

Half-line problem: $u(0, t) \leq 0$

Eigenfunctions:

$$\hat{\Phi} = I + \int_{(x^*, t^*)}^{(x, t)} e^{-ik(p(x, t, k) - p(y, \tau, k)) \hat{\sigma}^3} (U \hat{\Phi} dy + V \hat{\Phi} d\tau),$$

where $(x^*, t^*) \in \{(\infty, 0), (0, 0), (0, \infty)\}$,

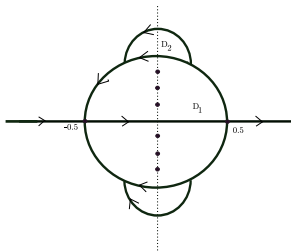
$$p(x, t, k) = \int_0^x \sqrt{m(\xi, t) + 1} d\xi - \int_0^t u(0, \tau) \sqrt{m(0, \tau) + 1} d\tau + \frac{t}{2\lambda}$$

Outline:

- ▷ $u(x, 0) = u_0(x) \mapsto \{a(k), b(k)\}$ (scattering for x -equ. from Lax pair)
- ▷ $\begin{pmatrix} u(0, t) = v_0(t) \\ u_x(0, t) = v_1(t) \\ u_{xx}(0, t) = v_2(t) \end{pmatrix} \mapsto \{A(k), B(k)\}$ (scattering for t -equ.)
- ▷ $\{a(k), b(k); A(k), B(k)\} \mapsto$ **RHP**
- ▷ compatibility of $\{u_0, v_0, v_1, v_2\}$: $A(k)b(k) - B(k)a(k) = 0, k \in D$

Half-line problem: RHP

$$D = \mathbb{C}_+ \setminus D_1$$



Define $d(k) = a(k)\bar{A}(\bar{k}) - b(k)\bar{B}(\bar{k})$; $R(k) = \frac{\bar{B}(\bar{k})}{a(k)d(k)}$, $k \in D_1$

- Residue conditions: at **zeros of $d(k)$** : \hat{v}_j, \hat{y}_j

- Jump matrix:

- ▶ $\begin{pmatrix} 1 & 0 \\ R(k)e^{2ikp} & 1 \end{pmatrix}$, $|k| = \frac{1}{2}$, $\text{Im } k > 0$

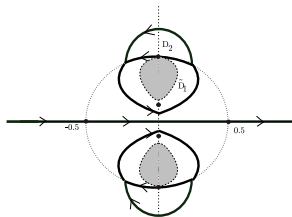
- ▶ $\begin{pmatrix} 1 & \bar{r}(k)e^{-2ikp} \\ -r(k)e^{2ikp} & 1 - |r|^2 \end{pmatrix}$, $|k| > \frac{1}{2}$, $\text{Im } k = 0$

- ▶ $\begin{pmatrix} 1 & 0 \\ (R-r)e^{2ikp} & 1 \end{pmatrix} \begin{pmatrix} 1 & (\bar{r} - \bar{R})e^{-2ikp} \\ 0 & 1 \end{pmatrix}$, $|k| < \frac{1}{2}$, $\text{Im } k = 0$

Contour deformations

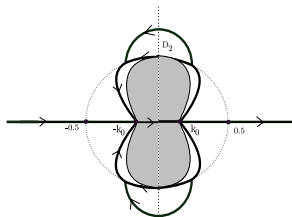
Initial contour deformation: according to signature table.

(i) $x/t > 2$



Soliton parameters: $\{\nu_j, \gamma_j\} \mapsto \{\hat{\nu}_j, \hat{\gamma}_j\}$

(ii) $0 < x/t < 2$



Parameters in asymptotic formulas: $r(k) \mapsto r(k) - R(k)$.

Completely integrable case: $u(0, t) = 0$

- **symmetry**: $A(k) = A\left(\frac{1}{4k}\right)$, $B(k) = B\left(\frac{1}{4k}\right)$
- global relation: $A(k)b(k) - B(k)a(k) = 0$, $\text{Im } k > 0$, $|k| > \frac{1}{2}$

allow solving $A(k)$, $B(k)$ in terms of $a(k)$, $b(k)$

- $R(k) = \frac{\bar{b}(1/4\bar{k})}{a(k)\Delta(k)}$,
where $\Delta(k) = a(k)\bar{a}\left(\frac{1}{4k}\right) - b(k)\bar{b}\left(\frac{1}{4k}\right)$
- residue conds.: at **zeros** $\{\lambda_j\}$ of $\Delta(k)$:

$$\text{Res}_{k=\lambda_j} \mu_1(y, t, k) = \frac{\bar{b}(1/4\bar{\lambda}_j)}{a(\lambda_j)\dot{\Delta}(\lambda_j)} e^{2i\lambda_j p} \mu_2(y, t, \lambda_j)$$