Liouville type theorems for anisotropic degenerate elliptic problems

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joint work with:

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Anisotropic operators on strips

- $S = \mathbb{R}^N \times (-1, 1)$, $x = (x', \lambda) \in S$

$$\mathcal{L}_{1,2}\ \sigma := \partial_i \left(A_{i,j}(1 - |\lambda|)\partial_j \sigma\right) + \partial_\lambda \left(A_{N,N}(1 - |\lambda|)^3 \partial_\lambda \sigma\right) +$$

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A being a uniformly elliptic matrix with bounded and measurable coefficients $i, j = 1 \cdots N - 1$
If A=Identity then

$$\mathcal{L}_{1,2} = \mathcal{L}_0 = (1 - |\lambda|)\Delta_{x'} \sigma + \partial_\lambda \left((1 - |\lambda|)^3 \partial_\lambda \sigma\right).$$

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- Filippas-M-Tertikas (2007) isotropic case
  $div(d^\alpha(x, \partial\Omega)\nabla\sigma)$, $\alpha \geq 1$, where Ω is a bounded C^2 domain

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Are the only bounded solutions σ of $\mathcal{L}_{1,2}$ in S the constants?

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If $\exists$ *a positive unbounded* $C^{0,1}$ *function* $\varphi(x', \lambda)$ *such that*

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- *for some $m, \tilde{m} > 0$*

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- $$\frac{R^2}{R^{2\tilde{m}}} \frac{\int_{L_R} \left(|\nabla_{x'}\varphi|^2 + (1-|\lambda|)^2(\partial_\lambda\varphi)^2\right)(1-|\lambda|)dx'd\lambda}{\int_{L_R}(1-|\lambda|)dx'd\lambda} \to 0$$

then the Liouville theorems holds true.

Applications

- First: The constant coefficients case The Liouville theorems applies to $\mathcal{L}_{1,2}$ for any $N \geq 2$ if $A_{i,N}$ does not depend on x_i and $A_{N,N}$ does not depend on λ, thus in particular to $\mathcal{L}_0$

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- Sketch of the proof. Take $\varphi = (\partial_N u)^{-1}$ and use 1-d hodograph transformation.

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- Sharpness of the Sobolev inequality: counterexample to the validity of the Sobolev inequality on all $\{|\lambda| < 1\}$ Filippas-M-Tertikas (2008)
- Counterexample to the validity of a global Harnack inequality on L_R
- Another approach: Bernstein gradient estimates give similar results but they only apply for $A = A(\lambda)$.

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\+ mixed terms
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$L_R := \{|x'| < R, \varphi < \gamma(R)\}$ for some positive nondecreasing function $\gamma(R)$ tending to $+\infty$ as $R \to +\infty$.

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- *(almost-supersolution property),*
- *(doubling)*
- *(weighted Sobolev inequality on L_R)*
- *(weighted Poincaré inequality on L_R)*
- *(balancing property).*

Consequences and developements

- The variable coefficients case The Liouville theorems applies to $\mathcal{L}_{1,2}$ for any $N \geq 2$ if $A_{i,N} = 0$ and $A_{N,N}(1 - |\lambda|)^{2-\epsilon}$ is a decreasing function of λ, for some $\epsilon \in (0, 2]$

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- multi-codimensional sets ie $S \to \mathbb{R}^{N-k} \times \Omega$ for some bounded smooth $\Omega \subset \mathbb{R}^k$