

Liouville type theorems for anisotropic degenerate elliptic problems

Luisa Moschini

University of Rome "La Sapienza"

joint work with:

Xavier Cabré, University Politecnica de Catalunya
ICREA, Barcelona, Spain

Anisotropic operators on strips

► $S = \mathbb{R}^N \times (-1, 1)$, $x = (x', \lambda) \in S$

$$\begin{aligned} \mathcal{L}_{1,2} \sigma &:= \partial_i (A_{i,j}(1 - |\lambda|) \partial_j \sigma) + \partial_\lambda (A_{N,N}(1 - |\lambda|)^3 \partial_\lambda \sigma) + \\ &+ \partial_\lambda (A_{N,j}(1 - |\lambda|)^2 \partial_j \sigma) + \partial_i (A_{i,N}(1 - |\lambda|)^2 \partial_\lambda \sigma) \end{aligned}$$

A being a uniformly elliptic matrix with bounded and measurable coefficients $i, j = 1 \dots N - 1$

If $A = \text{Identity}$ then

$$\mathcal{L}_{1,2} = \mathcal{L}_0 = (1 - |\lambda|) \Delta_{x'} \sigma + \partial_\lambda \left((1 - |\lambda|)^3 \partial_\lambda \sigma \right).$$

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- ▶ **degeneration** in terms of the **distance to set** $\{|\lambda| = 1\}$

- ▶ **Filippas-M-Tertikas (2007) isotropic case**

$\text{div}(d^\alpha(x, \partial\Omega) \nabla \sigma)$, $\alpha \geq 1$, where Ω is a bounded C^2 domain

Liouville type operators

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▶ for some $m, \tilde{m} > 0$

$$L_R := \{|x'| < R, \varphi < R^{\tilde{m}}\} \subseteq \{|x'| < R, |\lambda| < 1 - R^{-(1+m)}\}$$

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$$\frac{R^2 \int_{L_R} (|\nabla_{x'}\varphi|^2 + (1 - |\lambda|)^2 (\partial_\lambda \varphi)^2) (1 - |\lambda|) dx' d\lambda}{R^{2\tilde{m}} \int_{L_R} (1 - |\lambda|) dx' d\lambda} \rightarrow 0$$

then the Liouville theorems holds true.

Applications

- ▶ **First: The constant coefficients case** The Liouville theorem applies to $\mathcal{L}_{1,2}$ for any $N \geq 2$ if $A_{i,N}$ does not depend on x_i and $A_{N,N}$ does not depend on λ , thus in particular to \mathcal{L}_0

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- ▶ Sketch of the proof. Take $\varphi = (\partial_N u)^{-1}$ and use 1-d hodograph transformation.

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- ▶ Counterexample to the validity of a global Harnack inequality on L_R
- ▶ Another approach: Bernstein gradient estimates give similar results but they only apply for $A = A(\lambda)$.

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+ mixed terms

$w_i(x', \lambda) > 0$ in S , $w_i(x', \lambda) \in L_{loc}^\infty(S)$ and

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- ▶ *(balancing property).*

Consequences and developments

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- ▶ multi-codimensional sets ie $S \rightarrow \mathbb{R}^{N-k} \times \Omega$ for some bounded smooth $\Omega \subset \mathbb{R}^k$