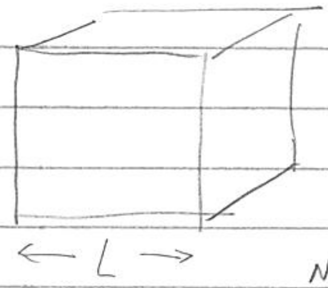
M. Grillakis



$B_L = [0, L]^3$, $|B_L| = L^3$

periodic BC

$$H_N = \sum_{a=1}^{N} -\Delta_{x_a} + \frac{1}{2}\sum_{a\neq b} v(x_a - x_b) \quad , \quad a = 1, \dots, N$$

$\psi(x_1, \dots, x_N)$ static $\quad E_N = \min_{\|\psi\|=1} (\psi, H_N \psi)$

thermodynamic limit

$$\lim_{N\to\infty} \frac{E_N}{N} = \mathcal{E}(\rho) \qquad \text{with} \quad v \geq 0$$

$$\int v < +\infty$$

$\frac{N}{|B_L|} = \rho \ll 1$ fixed

Mean-field $\quad$ prob. $F(x_1, \dots, x_N) \underset{N>>}{\approx} \prod_{a=1}^{N} f(x_a)$ (propagation of chaos

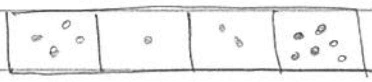
f prob density of "typical" particle

eq. for $f(t,x)$ (good for measurements)

Bose: quantized "photons", Einstein (massive particles)

- non interacting particles

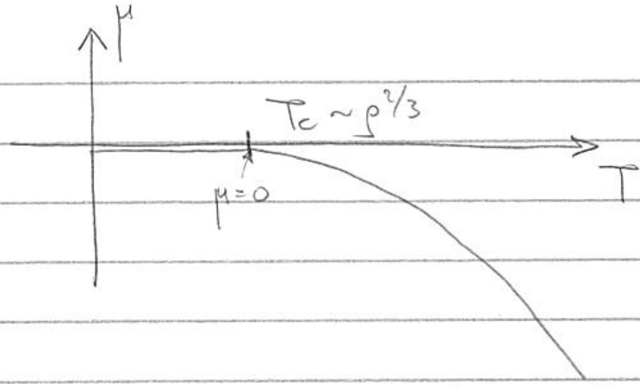
particles, states, $\frac{e^{ip\cdot x}}{\sqrt{|B_L|}}$, N particles, M states, ..



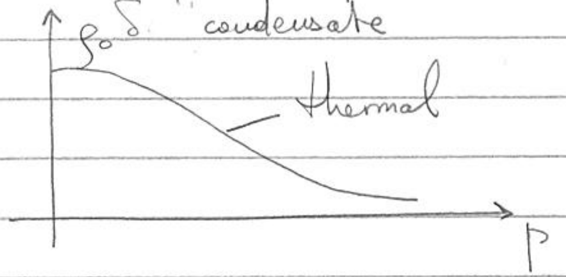
Bose-Einstein statistics

$$f_{BE}(\varepsilon) = \frac{1}{e^{-\beta(\varepsilon-\mu)} - 1} \quad , \quad \varepsilon_p = \frac{\hbar^2 |p|^2}{2\pi m} \quad , \quad \hbar = 2m = 1$$

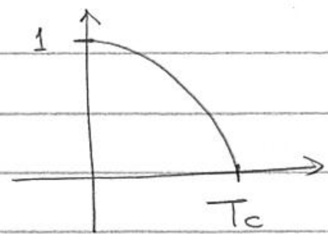
μ: chemical potential, $\mu = -\frac{3}{2} k_B T \log\left(\frac{m k_B T}{2\pi \hbar^2 \rho^{2/3}}\right)$



macroscopic occupation of $\vec{p}=0$ " $\varepsilon=0$

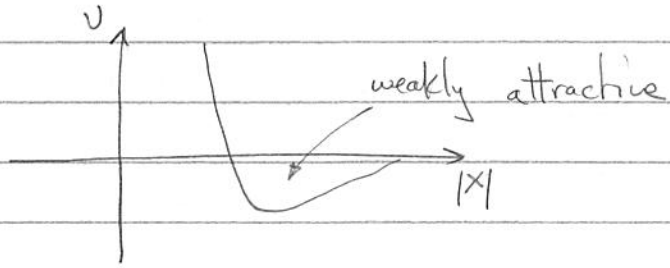


$$\zeta = \frac{\rho_0}{\rho} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$



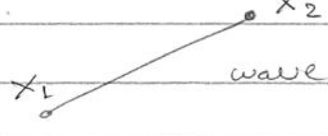
<u>Superfluids</u> He-4

(interacting particles)



two-fluid model : ρ normal, ρ superfluid $\approx 10\%$

two particle correlations



wave function

$$\psi(|x_1 - x_2|) = \psi(r) \quad , \quad \lim_{r\to+\infty} \psi(r) = \psi_\infty \neq 0$$

OLRDO

Interacting particles $\mathcal{E}(\rho)$ = expansion in ρ

Dyson, Lee-Huang-Yang, Bogoliubov

$$\mathcal{E}(\rho) \le 4\pi a \rho + \underset{\uparrow \atop \frac{128}{15\sqrt{\pi}}}{\mathbb{C}}(\rho a)^{3/2} + \mathbb{C}\, \rho^2 \log \rho + \cdots \quad '1960$$

upper bound

$a =$ scattering length

$$-\Delta w + \frac{1}{2} v w = 0$$

$$\lim_{x \to \infty} w = 1 \quad , \quad w \approx 1 - \frac{a}{|x|}$$

Lieb – Seiringer proved lower bound '2000

Schlein, H.T. Yau : rigorous derivation

M. Machedon, Elif Kuz, Grillakis :

<u>2nd quantization</u> $\mathbb{F}$ = Fock space (momentum representation)

creation $\leftarrow a_p^+ = \int dx \, \{ \bar{e}_p(x) \, a_x^+ \}$ $\quad e_p(x) = \dfrac{e^{ix \cdot p}}{\sqrt{|B_L|}}$

annihilation $\leftarrow a_p = \int dx \, \{ e_p(x) \, a_x \}$

$$[a_{p_1}, a_{p_2}^+] = \delta(p_1 - p_2)$$

Fock Hamiltonian $\mathcal{H}_N = \sum_p |p|^2 a_p^+ a_p + \dfrac{1}{2|B_L|} \sum_{p_1+p_2=p_3+p_4} \underbrace{a_{p_1}^+ a_{p_2}^+}_{Q^+} \hat{v}(p_1 - p_3) \underbrace{a_{p_3} a_{p_4}}_{Q}$

$|\psi\rangle$ = Fock vector

$E = \langle \psi | \mathcal{H}_N | \psi \rangle$ $\quad$ Energy $\min_{\substack{\langle \psi | \psi \rangle = 1 \\ \langle \psi | \mathcal{N} | \psi \rangle = N}} \langle \psi | \mathcal{H}_N | \psi \rangle$

$\mathcal{N} = \sum_p a_p^+ a_p$ number operator

$$\hat{v}_p = \int dx \left\{ e^{ixp} \; v(x) \right\}$$

<u>Trial state</u> ? Evolution equ ?

$$\mathcal{A}(t) = \sqrt{N} \left(-z a_0^+ + \bar{z} a_0 \right) \qquad p=0$$

$$|z|^2 = \text{condensate fraction}$$

$$\mathcal{B}(z) = \frac{1}{2} \sum_p \left(-k_p a_{-p}^+ a_p^+ + \bar{k}_p a_{-p} a_p \right) \qquad k_p = ? \quad k_0 = 0$$

$$\mathcal{N}(i\theta) = -\frac{i}{2} \sum_p \theta_p \left(a_p a_p^+ + a_p^+ a_p \right) \qquad \theta = \text{phase} = ?$$

State $|\psi_{app}\rangle = e^{-\mathcal{A}} e^{-\mathcal{N}} e^{-\mathcal{B}} |0\rangle$

vacuum $|0\rangle = (1, 0, \cdots)$

$e^{-\mathcal{A}}|0\rangle$ creates $N|z|^2$ particles at $p=0$

$e^{-\mathcal{N}} e^{-\mathcal{B}}$ creates pairs $\rightarrow \begin{pmatrix} \chi_p & \bar{\phi}_p \\ \phi_p & \bar{\chi}_p \end{pmatrix}$

$$|\chi_p|^2 - |\phi_p| = 1, \quad \forall p \quad \left(\begin{array}{c}\text{use Lie-} \\ \text{algebra isom.}\end{array}\right)$$

$p = \frac{n}{2\pi L}$, $n \in \mathbb{Z}^3$

$\mathcal{A}$ = "base" condensate $\rightarrow$ end: "dressed" condensate.

conjecture: this is a "good" approximation.

Take thermodynamic limit p = continuous

$$\Theta_p = \rho z^2 \delta_p + \underbrace{\chi_p \phi_p}_{\text{interaction term}}$$

$$\mathcal{E} = \frac{\rho}{2}\,\hat{v}_0\left(1-|z|^4\right) + \frac{1}{2\rho}\int\frac{dp_1\,dp_2}{(2\pi)^6}\left\{\overline{\ominus}_{p_2}\,\hat{v}_{p_2-p_1}\,\overline{\ominus}_{p_1}\right\}$$

$$+\frac{1}{2\rho}\int\frac{dp}{(2\pi)^3}\,\varepsilon_p\left(|\chi_p|^2+|\psi_p|^2-1\right)+\frac{1}{2\rho}\int\frac{dp_1\,dp}{(2\pi)^6}\,|\phi_{p_2}|^2\,\hat{v}_{p_2-p_1}\,|\phi_{p_1}|^2$$

$$\varepsilon_p = |p|^2 + \rho|z|^2\,\hat{v}_p$$

$$\hat{v}_0 = \int v$$

$$\hat{\psi}_p = -2\chi_p\phi_p\,, \qquad \hat{\tau}_p = |\chi_p|^2+|\phi_p|^2-1 = \sqrt{1+|\psi_p|^2}-1$$

$$\left(\varepsilon_p+\Gamma_0+v_p-v_0\right)\hat{\psi}_p + \left(\Gamma_p-\rho z^2\,\hat{v}_p\right)\left(\hat{\tau}_p+1\right)=0$$

$$\Gamma_p = -\int\hat{v}_{p-p'}\,\chi_p\,\phi_p = \frac{1}{2}\,\hat{v}*\hat{\psi}\,, \qquad \Gamma_0=\frac{1}{2}\langle\psi, v\rangle$$

$$\frac{\hat{\psi}_p}{\hat{\tau}_p+1} = \frac{\rho z^2\,\hat{v}_p-\Gamma_p}{\varepsilon_p+\Gamma_0+V_1-V_0}$$

$\underline{\underline{p=0}}$

$$\frac{\hat{\psi}_0}{\sqrt{1-|\psi_0|^2}} = \frac{\rho z^2\,\hat{v}_0-\Gamma_0}{\rho z^2\,\hat{v}_0+\Gamma_0}$$

$\hat{\psi}_0$ finite $\quad \int\psi < +\infty$

$$|p|^2\hat{\psi}_p + \Gamma_p = \rho\,\hat{v}_p \quad \Leftarrow \quad -\Delta\psi+\frac{1}{2}v\psi = v\rho$$

$$\Leftrightarrow\quad -\Delta\left(\psi-2\rho\right)+\frac{1}{2}v\left(\psi-2\rho\right)=0$$

$$\psi \approx \frac{2\rho}{|X|}$$

$$\hat{\psi} = \frac{2A\rho}{|p|^2} \qquad \xi = |z|^2 \quad \text{condensate fraction}$$

$$1-|z|^2 = \text{pair fraction}$$

$$\approx C\sqrt{\rho}$$

$$\Gamma_0 = \text{important} = \frac{1}{2}\langle v, \psi\rangle \approx \rho\,(\underbrace{\hat{v}_0 - 8\pi a}_{\mu > 0})$$

$$\psi \approx \frac{e^{-\sqrt{\mu\rho}\,|x_1-x_2|}}{|x_1-x_2|} \quad \leftarrow \text{subtract (dress) average}$$