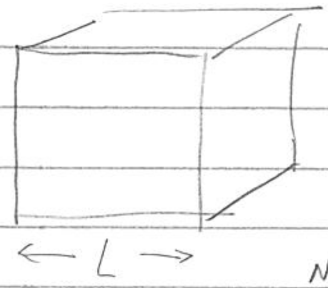
M. Grillakis



$B_L = [0, L]^3$, $|B_L| = L^3$

periodic BC

$$H_N = \sum_{a=1}^{N} -\Delta_{x_a} + \frac{1}{2} \sum_{a \neq b} v(x_a - x_b) \quad , \quad a = 1, \dots, N$$

$\psi(x_1, \dots, x_N)$ static $\qquad E_N = \min_{\|\psi\|=1} (\psi, H_N \psi)$

thermodynamic limit

$$\lim_{N \to \infty} \frac{E_N}{N} = \mathcal{E}(\rho) \qquad \text{with} \quad v \geq 0$$

$$\int v < +\infty$$

$$\frac{N}{|B_L|} = \underset{\text{fixed}}{\rho} \ll 1$$

Mean-field $\quad$ prob. $\underset{N \gg}{F(x_1, \dots, x_N)} \approx \prod_{a=1}^{N} f(x_a)$ (propagation of chaos

f prob density of "typical" particle

eq. for $f(t,x)$ (good for measurements)

Bose : quantized "photons" , Einstein (massive particles)

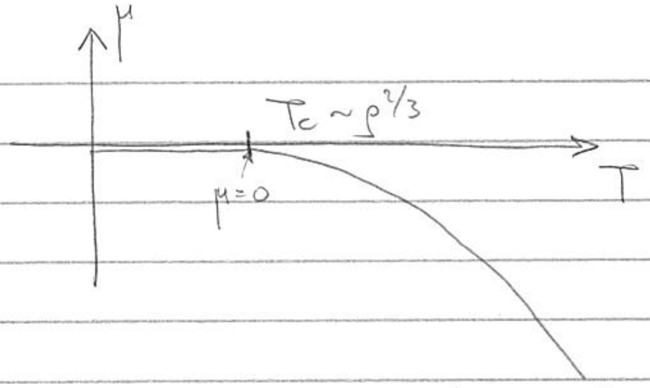
- non interacting particles

particles, states, $\frac{e^{ip \cdot x}}{\sqrt{|B_L|}}$, N particles, M states, ..

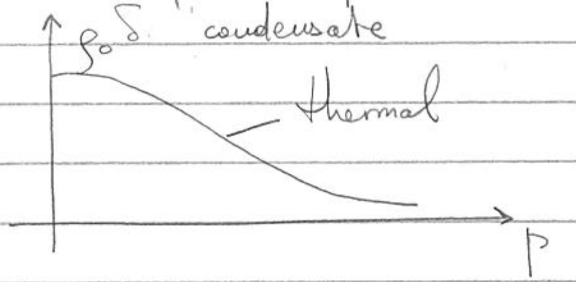
Bose-Einstein statistics

$$f_{BE}(\varepsilon) = \frac{1}{e^{-\beta(\varepsilon - \mu)} - 1} \quad , \quad \varepsilon_p = \frac{\hbar^2 |p|^2}{2\pi m} \quad , \quad \hbar = 2m = 1$$

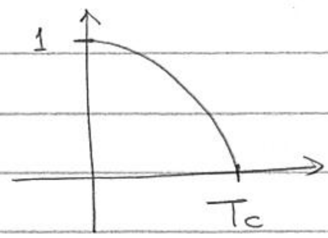
μ: chemical potential , $\mu = -\frac{3}{2} K_B T \log \left(\frac{m k_B T}{2\pi \hbar^2 \rho^{2/3}} \right)$



macroscopic occupation of $\vec{p} = 0$ " $\varepsilon = 0$

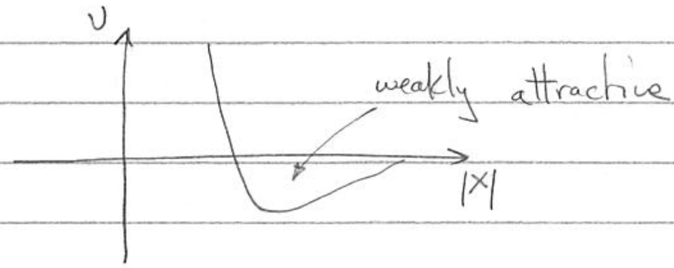


$$\xi = \frac{\rho_0}{\rho} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$



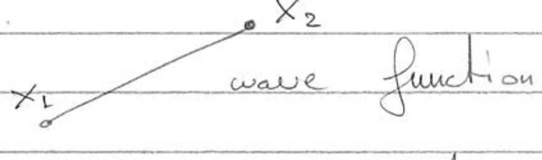
<u>Superfluids</u> He-4

(interacting particles)



two-fluid model : ρ normal, ρ superfluid $\approx 10\%$

two particle correlations



$$\psi(|x_1 - x_2|) = \psi(r), \quad \lim_{r \to +\infty} \psi(r) = \psi_\infty \neq 0$$

OLRDO

Interacting particles $\mathcal{E}(\rho)$ = expansion in ρ

Dyson, Lee-Huang-Yang, Bogoliubov

$$\mathcal{E}(\rho) \le 4\pi a\rho + \underset{\uparrow}{\mathbb{C}}(\rho a)^{3/2} + \mathbb{C}\,\rho^2 \log \rho + \cdots \quad '1960$$

$\frac{128}{15\sqrt{\pi}}$ upper bound

a = scattering length

$$-\Delta w + \frac{1}{2} v w = 0$$

$$\lim_{x \to \infty} w = 1 \quad , \quad w \approx 1 - \frac{a}{|x|}$$

Lieb – Seiringer proved lower bound '2000

Schlein, H.T. Yau : rigorous derivation

M. Machedon, Elif Kuz, Grillakis :

<u>2nd quantization</u> $\mathbb{F}$ = Fock space (momentum representation)

creation $\leftarrow a_p^+ = \int dx \, \{ \bar{e}_p(x) \, a_x^+ \}$ $\quad e_p(x) = \frac{e^{ix \cdot p}}{\sqrt{|B_L|}}$

annihilation $\leftarrow a_p = \int dx \, \{ e_p(x) \, a_x \}$

$$[a_{p_1}, a_{p_2}^+] = \delta(p_1 - p_2)$$

Fock Hamiltonian $\mathcal{H}_N = \sum_p |p|^2 a_p^+ a_p + \frac{1}{2|B_L|} \sum_{p_1+p_2=p_3+p_4} \underbrace{a_{p_1}^+ a_{p_2}^+}_{Q^+} \hat{v}(p_1-p_3) \underbrace{a_{p_3} a_{p_4}}_{Q}$

$|\psi\rangle$ = Fock vector

$E = \langle \psi | \mathcal{H}_N | \psi \rangle$ $\quad$ Energy $\min\limits_{\substack{\langle\psi|\psi\rangle=1 \\ \langle\psi|\mathcal{N}|\psi\rangle = N}} \langle \psi | \mathcal{H}_N | \psi \rangle$

$$\mathcal{N} = \sum_p a_p^+ a_p \quad \text{number operator}$$

$$\hat{v}_p = \int dx \left\{ e^{ixp} \, v(x) \right\}$$

<u>Trial state</u> ? Evolution equ ?

$$\mathcal{A}(t) = \sqrt{N} \left(-z a_0^+ + \bar{z} a_0 \right) \qquad p=0$$

$$|z|^2 = \text{condensate fraction}$$

$$\mathcal{B}(z) = \frac{1}{2} \sum_p \left(-k_p a_{-p}^+ a_p^+ + \bar{k}_p a_{-p} a_p \right) \qquad k_p = ? \quad k_0 = 0$$

$$\mathcal{N}(i\theta) = -\frac{i}{2} \sum_p \theta_p \left(a_p a_p^+ + a_p^+ a_p \right) \qquad \theta = \text{phase} = ?$$

State $|\psi_{app}\rangle = e^{-\mathcal{A}} e^{-\mathcal{N}} e^{-\mathcal{B}} |0\rangle$

vacuum $|0\rangle = (1, 0, \cdots)$

$e^{-\mathcal{A}}|0\rangle$ creates $N|z|^2$ particles at $p=0$

$e^{-\mathcal{N}} e^{-\mathcal{B}}$ creates pairs $\rightarrow \begin{pmatrix} \chi_p & \bar{\phi}_p \\ \phi_p & \bar{\chi}_p \end{pmatrix}$

$$|\chi_p|^2 - |\phi_p| = 1, \quad \forall p \qquad \text{(use Lie-algebra isom.)}$$

$$p = \frac{n}{2\pi L}, \quad n \in \mathbb{Z}^3$$

$\mathcal{A}$ = "base" condensate $\rightarrow$ end: "dressed" condensate.

conjecture: this is a "good" approximation.

Take thermodynamic limit p = continuous

$$\underbrace{\textcircled{H}_p = \rho z^2 \delta_p + \chi_p \phi_p}_{\text{interaction term}}$$

$$\mathcal{E} = \frac{\rho}{2}\,\hat{v}_0\left(1-|z|^4\right) + \frac{1}{2\rho}\int \frac{dp_1\,dp_2}{(2\pi)^6}\left\{ \overline{\Theta}_{p_2}\,\hat{v}_{p_2-p_1}\,\overline{\Theta}_{p_1} \right\}$$

$$+\frac{1}{2\rho}\int \frac{dp}{(2\pi)^3}\,\varepsilon_p\left(|\chi_p|^2+|\psi_p|^2-1\right) + \frac{1}{2\rho}\int \frac{dp_1\,dp}{(2\pi)^6}\,|\phi_{p_2}|^2\,\hat{v}_{p_2-p_1}\,|\phi_{p_1}|^2$$

$$\varepsilon_p = |p|^2 + \rho|z|^2\,\hat{U}_p$$

$$\hat{v}_0 = \int v$$

$$\hat{\psi}_p = -2\,\chi_p\,\phi_p\,, \qquad \hat{\tau}_p = |\chi_p|^2 + |\phi_p|^2 - 1 = \sqrt{1+|\psi_p|^2} - 1$$

$$\left(\varepsilon_p + \Gamma_0 + v_p - v_0\right)\hat{\psi}_p + \left(\Gamma_p - \rho z^2\,\hat{U}_p\right)\left(\hat{\tau}_p+1\right) = 0$$

$$\Gamma_p = -\int \hat{U}_{p-p'}\,\chi_p\,\phi_p = \frac{1}{2}\,\hat{v} * \hat{\psi}\,, \qquad \Gamma_0 = \frac{1}{2}\langle \psi, v\rangle$$

$$\frac{\hat{\psi}_p}{\hat{\tau}_p+1} = \frac{\rho z^2\,\hat{U}_p - \Gamma_p}{\varepsilon_p + \Gamma_0 + V_1 - V_0}$$

$\underline{\underline{p=0}}$

$$\frac{\hat{\psi}_0}{\sqrt{1-|\psi_0|^2}} = \frac{\rho z^2\,\hat{U}_0 - \Gamma_0}{\rho z^2\,\hat{v}_0 + \Gamma_0}$$

$\hat{\psi}_0$ finite $\quad \int \psi < +\infty$

$$|p|^2\hat{\psi}_p + \Gamma_p = \rho\,\hat{v}_p \;\Leftarrow\; -\Delta\psi + \frac{1}{2}v\psi = v\rho$$

$$\Leftrightarrow\; -\Delta\left(\psi - 2\rho\right) + \frac{1}{2}v\left(\psi - 2\rho\right) = 0$$

$$\psi \approx \frac{2\rho}{|x|}$$

$$\hat{\psi} = \frac{2A\rho}{|p|^2}$$

$\xi = |z|^2$ condensate fraction

$1 - |z|^2 =$ pair fraction

$\approx C\sqrt{\rho}$

$$\Gamma_0 = \text{important} = \frac{1}{2}\langle v, \psi\rangle \approx \rho\,\underbrace{(\hat{v}_0 - 8\pi a)}_{\mu > 0}$$

$$\psi \approx \frac{e^{-\sqrt{\mu\rho}\,|x_1 - x_2|}}{|x_1 - x_2|}$$ ← subtract (dress) average