Department of Applied Mathematics

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June 27, 2013

On the asymptotic behavior of symmetric solutions of the vector Allen-Cahn equation

Joint work with N.Alikakos

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Motivation

Assume $W : \mathsf{R}^m \to \mathsf{R}$ is invariant under a finite reflection group G and has a unique nondegenerate zero $a \in F$, (F is a fundamental region for the action of G on R^m).

Motivation

Assume $W : \mathsf{R}^m \to \mathsf{R}$ is invariant under a finite reflection group G and has a unique nondegenerate zero $a \in F$, (F is a fundamental region for the action of G on R^m).

Theorem Then the vector Allen-Cahn equation

$$\Delta u = W_u(u), \quad x \in \mathsf{R}^n,$$

admits an entire equivariant solutions $u : \mathsf{R}^n \to \mathsf{R}^m$ that *connects* the $|G|$ minima of W and is such that

$$|u(x) - a| \leq K e^{-kd(x,\partial F)}, x \in F.$$

(Bronsard Gui Shatzmann; Gui Shatzmann; Alikakos+ F.)

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In particular, given a unit vector $\nu \in \mathsf{R}^n$ one may wonder about the existence of the limit

(2) $$\lim_{\lambda \to +\infty} u(x' + \lambda\nu) = \tilde{u}(x'),$$

where x' is the projection of x on the plane orthogonal to ν.

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where x' is the projection of x on the plane orthogonal to ν.

One can conjecture that this limit does indeed exist and that $\tilde{u}$ is a solution of the same system equivariant with respect to the subgroup $G_\nu \subset G$ that leave ν fixed.

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In particular, given a unit vector $\nu \in \mathsf{R}^n$ one may wonder about the existence of the limit

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where x' is the projection of x on the plane orthogonal to ν.

One can conjecture that this limit does indeed exist and that $\tilde{u}$ is a solution of the same system equivariant with respect to the subgroup $G_\nu \subset G$ that leave ν fixed.

The previous exponential estimate gives a positive answer to this conjecture for the case where ν is inside the set F.
(F is a fundamental region for the action of G on R^n. G acts both on the domain and on the target space).

In what follows we discuss how one can go one step forward.

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We shows that the conjecture is true when ν belongs to the interior of one of the walls of the set F above and G_ν is the subgroup of order two generated by the reflection with respect to that wall.

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We shows that the conjecture is true when ν belongs to the interior of one of the walls of the set F above and G_ν is the subgroup of order two generated by the reflection with respect to that wall.

Our approach has an abstract character and can probably be used to discuss the case where ν belongs to the intersection of two walls of F and the order of G_ν is $4, 6 \ldots$ depending on the structure of G.
We also believe that the process can be iterated to prove a whole hierarchy of limits corresponding to all possible choice of ν and always $\tilde{u}$ is a solution of the system equivariant with respect to the subgroup G_ν.

The problem

For $x \in \mathsf{R}^d$ let $\hat{x} = (-x_1, x_2, \ldots, x_d)$ the reflection in the plane $x_1 = 0$.

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$\mathbf{H}_2-$ There is a nondegenerate minimizer $a \in \mathsf{R}^m$ such that $a_1 > 0$ and

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Let $\Omega \subset \mathsf{R}^n$ a smooth (bounded or unbounded) domain which is *convex-symmetric* in the sense

$$(x_1, \ldots, x_n) \in \Omega \Rightarrow (tx_1, \ldots, x_n) \in \Omega, \quad |t| \leq 1.$$

We consider the problem

$$\mathbf{P}) \begin{cases} \Delta u = W_z(u), & x \in \Omega, \\ u(\hat{x}) = \hat{u}(x), & x \in \Omega. \end{cases}$$

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From the assumptions on W it follows the existence of a map $\bar{u} : \mathsf{R} \to \mathsf{R}^m$ such that

$$J_{\mathsf{R}}(\bar{u}) = \min J_{\mathsf{R}}(v); \quad J_{\mathsf{R}}(v) = \int_{\mathsf{R}} \frac{1}{2}|v'|^2 + W(v)$$

where the minimum is taken on the set of maps $v \in W^{1,2}_{\mathrm{Loc}}(\mathsf{R}; \mathsf{R}^m)$ that are symmetric $v(-s) = \hat{v}(s)$ and connect $\hat{a}$ to a.

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where the minimum is taken on the set of maps $v \in W^{1,2}_{\mathrm{Loc}}(\mathsf{R}; \mathsf{R}^m)$ that are symmetric $v(-s) = \hat{v}(s)$ and connect $\hat{a}$ to a.

We expect that problem **P)** has a solution u which away from $\partial\Omega$ is close to $\bar{u}$.

Theorem. A

Assume W and Ω as before and assume that $\bar{u}$ is unique and non degenerate in the sense that the operator T defined by

$$Tv = -v'' + W_{zz}(\bar{u}), \qquad D(T) = W_S^{1,2}(\mathbf{R};\mathbf{R}^m),$$

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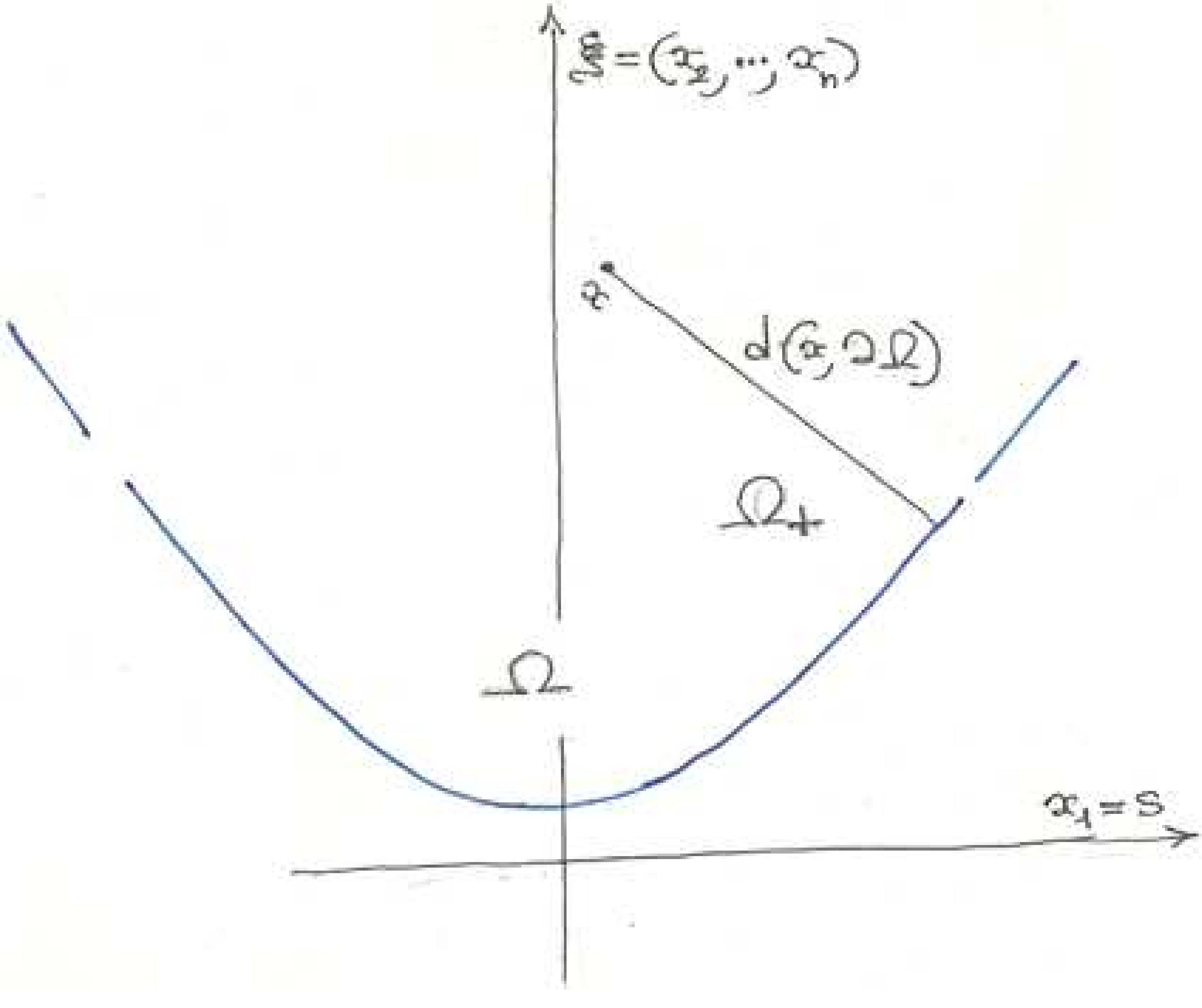
Then problem **P**) *has a bounded classical solution* u *such that*

(i) $u(\overline{\Omega_+}) \subset \overline{\mathbf{R}_+{}^m}$,

(ii) *There exist* $k, K > 0$ *such that*

$$|u(x) - \bar{u}(x_1)| \leq Ke^{-kd(x,\partial\Omega)}, \quad x \in \Omega.$$



x' = (x_2, ..., x_n)
x
d(x, ∂Ω)
Ω_+
Ω
x_1 = s

Proof of Theorem A. The case $\Omega = \mathsf{R} \times \omega$.

1. From the analysis in

G.F. Equivariant entire solutions of the elliptic system $\Delta u = W_u(u)$ for general $G-$ invariant potentials. (*CV PDE 2012*)

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1. From the analysis in

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it follows the existence of a symmetric solution $u : \Omega \to \mathsf{R}^m$ of problem **P**)
Moreover u is a **local minimizer** of the Allen-Cahn energy J in the set of symmetric maps and we have

$$\|u\|_{C^1(\Omega;\mathsf{R}^m)} \leq M$$

$$|u(s,\xi) - a| + |u_s(s,\xi)| \leq K_0 e^{-k_0 s}, \quad (s,\xi) \in \mathsf{R}_+ \times \omega.$$

where $x_1 = s,\ (x_2, \dots, x_n) = \xi$.

2. **Definition.** *A map* $u \in C^2(\overline{\Omega}; \mathbf{R}^m) \cap L^\infty(\Omega; \mathbf{R}^m)$ *is a local symmetric minimizer of the Allen-Cahn energy if*

$$J_A(u) \leq J_A(u + v), \qquad . \tag{5}$$

for all symmetric $v \in W_0^{1,2}(A; \mathbf{R}^m) \cap L^\infty(A; \mathbf{R}^m)$ *and for all smooth bounded open symmetric subset* $A \subset \Omega$.

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The Allen-Cahn energy is defined by

$$J_A(u) := \int_A \frac{1}{2}|\nabla u|^2 + W(u) \tag{8}$$

for every bounded domain $A \subset \Omega$.

The effective potential

If $\bar{u} + v$ satisfies the estimates in 1. then $v(\cdot, \xi)$ is bounded in $W_S^{1,2}(\mathsf{R}; \mathsf{R}^m)$ by some constant $C > 0$.

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Let
$$\mathcal{B}_C = \{v \in W_S^{1,2}(\mathsf{R}; \mathsf{R}^m) : \|v\|_1 < C\}$$
where $\|\cdot\|_1$ is the $W^{1,2}$ norm,
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where $\|\cdot\|_1$ is the $W^{1,2}$ norm,
let $\langle\cdot,\cdot\rangle$ the inner product in $L^2(\mathsf{R};\mathsf{R}^m)$ and $\|\cdot\|$ the corresponding norm.

For $v\in\mathcal{B}_C$ we define

$$\mathbf{e}(v)=\frac{1}{2}(\langle\bar{u}_s+v_s,\bar{u}_s+v_s\rangle-\langle\bar{u}_s,\bar{u}_s\rangle)+\int_{\mathsf{R}}(W(\bar{u}+v)-W(\bar{u})),$$

and let

$$\mathcal{S}=\{\nu\in W^{1,2}_S(\mathsf{R};\mathsf{R}^m):\|\nu\|=1\}.$$

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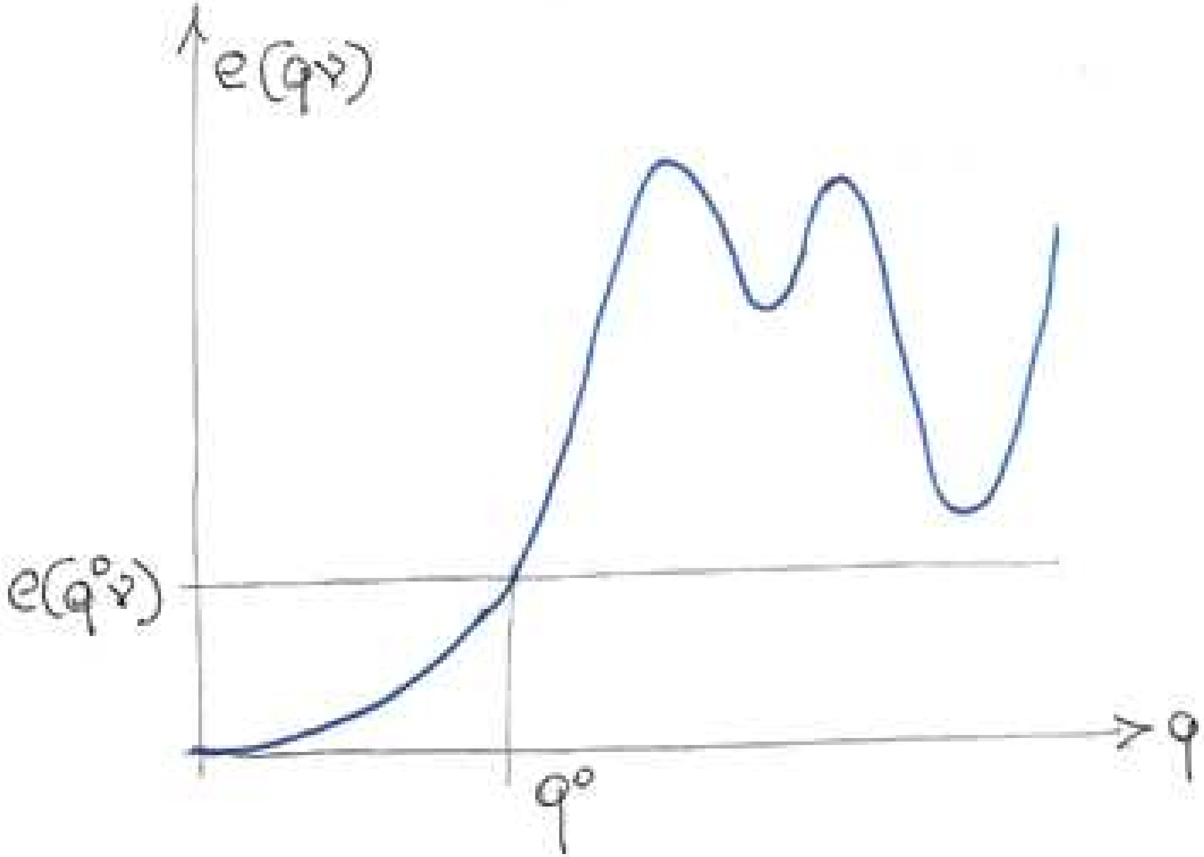
$$\mathbf{e}(q\nu) \geq \tilde{\mathbf{e}}(p, q, \nu) \quad \text{where}$$

$$\tilde{\mathbf{e}}(p, q, \nu) := \mathbf{e}(p\nu) + D_q\mathbf{e}(q\nu)(q - p),\ \ 0 \leq p \leq q \leq q^\circ,\ \nu \in \mathcal{S},$$


e(qν)
e(q°ν)
q°
q

Polar representation of the energy

3. Let $w : \mathsf{R} \times \omega \to \mathsf{R}^m$ a map that satisfies the estimates in 1. Define

$$q^w(\xi) = \|w(\cdot,\xi) - \bar{u}\|, \quad \nu^w(\xi) = \frac{w(\cdot,\xi) - \bar{u}}{\|w(\cdot,\xi) - \bar{u}\|}, \text{ if } q^w(\xi) > 0.$$

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Then, for $q^w \neq 0$, we have $w = \bar{u} + q^w \nu^w$.

Let $B_r = \{|\xi| < r\}$ and $\mathcal{C}_r = \mathsf{R} \times B_r$. Then we have:

$$J_{\mathcal{C}_r}(w) = \tfrac{1}{2} \textstyle\sum_j \int_{B_r} \langle w_{\xi_j}, w_{\xi_j} \rangle d\xi + \int_{B_r} \mathbf{e}(q^w \nu^w) d\xi + \bar{C},$$

$$= \textstyle\int_{B_r} \tfrac{1}{2}(|\nabla q^w|^2 + (q^w)^2 \sum_j \langle \nu^w_{\xi_j}, \nu^w_{\xi_j} \rangle) d\xi + \int_{B_r} \mathbf{e}(q^w \nu^w) d\xi + \bar{C}$$

Deformation Lemma

Given $0 < \bar{q} < q^\circ$ let $A_{\bar{q}}$ the set

$$A_{\bar{q}} := \{\xi : \|u(\cdot, \xi) - \bar{u}\| > \bar{q}\}.$$

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Fix $\rho > 0$ and let $\Sigma = A_{\bar{q}} \cap \{r < |\xi| < r + \rho\}$. Then there is a constat $C_1 > 0$ independent of r and a Lipschitz map v such that

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(iii) $q^v = \bar{q}$ on $\quad A_{\bar{q}} \cap \partial B_{r+\frac{\rho}{2}}$,

(iv) $J_{\mathcal{C}_{r+\rho}}(v) - J_{\mathcal{C}_{r+\rho}}(u) \leq C_1 \mathcal{H}^{n-1}(\Sigma)$.

Proof of Lemma 1

$$\text{Set}\quad v^{\bar{q}}(s,\xi) := \bar{u}(s) + \bar{q}\frac{u(s,\xi)-\bar{u}(s)}{\|u-\bar{u}\|},\ s \in \mathsf{R}, \xi \in \Sigma$$

and define, for $s \in \mathsf{R}$ and $\xi \in \Sigma$

$$v(s,\xi) := (1 - |1 - 2\frac{|\xi|-r}{\rho}|)v^{\bar{q}}(s,\xi) + |1 - 2\frac{|\xi|-r}{\rho}|u(s,\xi)$$

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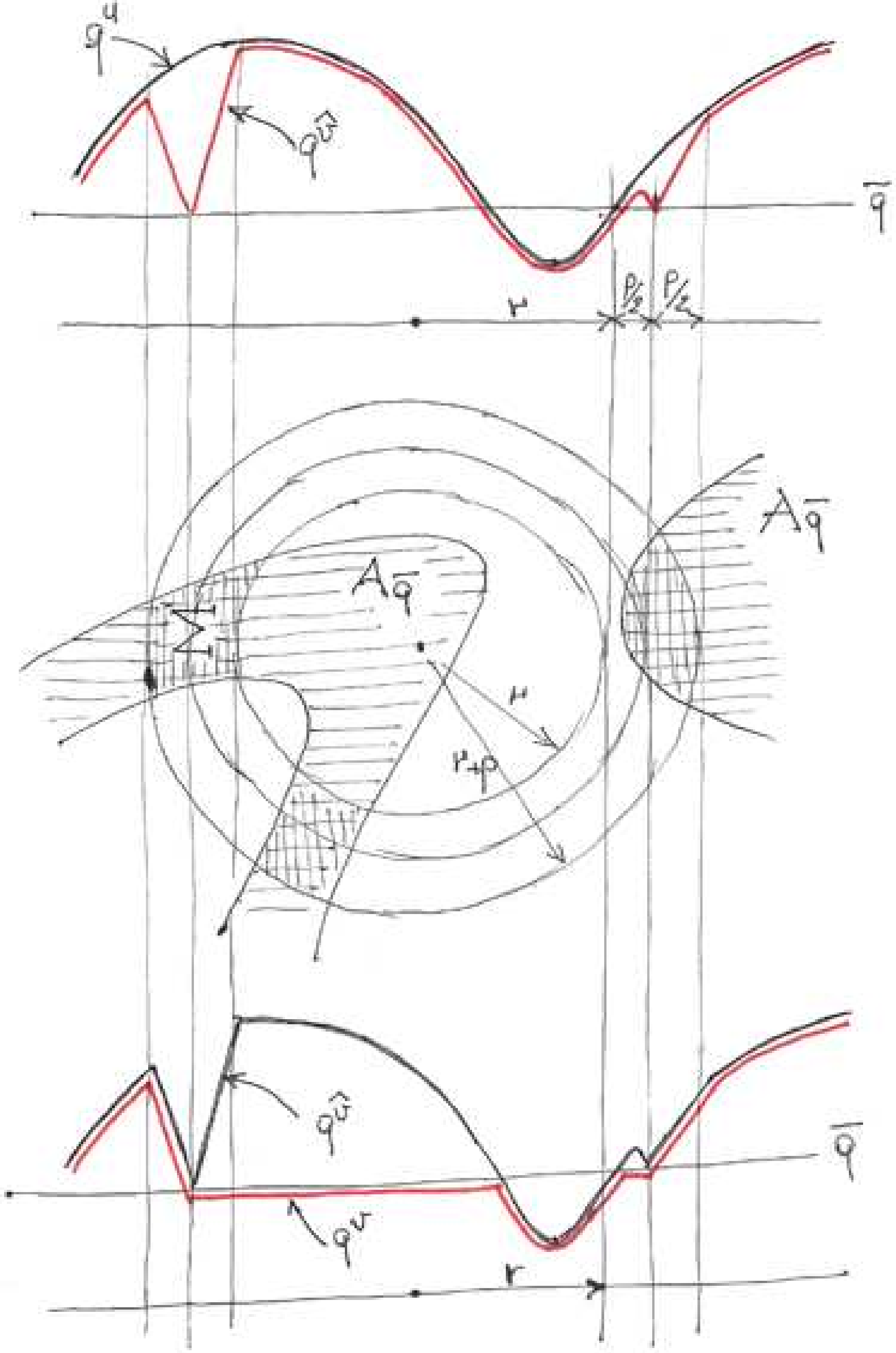
2. $|\xi| = r \Rightarrow v = u,$
3. $|\xi| = r + \rho \Rightarrow v = u,$
4. $|\xi| = r + \frac{\rho}{2} \Rightarrow v = v^{\bar{q}}.$



q^u
$q^{\hat{u}}$
$\bar{q}$
r
$\rho/2$
$\rho/2$
$A_{\bar{q}}$
$A_{\bar{q}}$
Σ
r
$r+\rho$
$q^{\hat{u}}$
$\bar{q}$
q^v
r

Replacement lemma

Lemma. 2 Let v as in Lemma **1** and let φ the solution of

$$\begin{cases} \Delta\varphi = c^2\varphi, & \text{in} \quad B_{r+\frac{\rho}{2}}, \\ \varphi = \bar{q}, & \text{in} \quad \partial B_{r+\frac{\rho}{2}}. \end{cases}$$

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(iii) $w = q^w \nu^v + \bar{u}$ on $\mathcal{C}_{r+\frac{\rho}{2}}$,

Replacement lemma

Lemma. 2 Let v as in Lemma **1** and let φ the solution of

$$\begin{cases} \Delta\varphi = c^2\varphi, & \text{in} \quad B_{r+\frac{\rho}{2}}, \\ \varphi = \bar{q}, & \text{in} \quad \partial B_{r+\frac{\rho}{2}}. \end{cases}$$

Then there is a constant $c_1 > 0$ independent of r and a Lipschitz map w with the following properties

(i) $w = v$ on $\mathsf{R} \times \omega \setminus \mathcal{C}_{r+\frac{\rho}{2}}$,

(ii) $q^w \leq \varphi \leq \bar{q}$ on $\mathcal{C}_{r+\frac{\rho}{2}}$,

(iii) $w = q^w \nu^v + \bar{u}$ on $\mathcal{C}_{r+\frac{\rho}{2}}$,

(iv) $J_{\mathcal{C}_{r+\frac{\rho}{2}}}(v) - J_{\mathcal{C}_{r+\frac{\rho}{2}}}(w) \geq c_1 \mathcal{H}^{n-1}(A_{\bar{q}} \cap B_r)$.

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$$\mathcal{J}(p) \;=\; \int_{I_b} \frac{1}{2}|\nabla p|^2 + \mathbf{e}(|p|\nu^v).$$

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Define w by

$$w = \begin{cases} v, \text{ for } (s,\xi) \in \mathsf{R} \times (\omega \setminus I_b), \\[2ex] \bar{u} + q^w \nu^v = \bar{u} + \min\{q^*, q^v\}\nu^w, \text{ for } \xi \in I_b. \end{cases}$$

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\begin{aligned}
J_{I_b \cap \{q^v > q^*\}}(v) - J_{I_b \cap \{q^v > q^*\}}(w) \geq \\
&\int_{I_b \cap \{q^v > q^*\}} \frac{1}{2}|\nabla q^v|^2 + \mathbf{e}(q^v \nu) \\
&- \int_{I_b \cap \{q^v > q^*\}} \frac{1}{2}|\nabla q^*|^2 + \mathbf{e}(q^* \nu) \\
&+ \int_{I_b \cap \{q^v > q^*\}} ((q^v)^2 - (q^*)^2) \sum_j \langle \nu^v_{\xi_j}, \nu^v_{\xi_j} \rangle \geq 0
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Quantitative estimate of $J_{I_b \cap \{q^v > q^*\}}(v) - J_{I_b \cap \{q^v > q^*\}}(w)$.

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$$J_{I_b\cap\{q^v>q^*\}}(v) - J_{I_b\cap\{q^v>q^*\}}(w) \geq \int_{I_b\cap\{q^v>q^*\}} \frac{1}{2}|\nabla q^v|^2 + \mathbf{e}(q^v\nu) - \int_{I_b\cap\{q^v>q^*\}} \frac{1}{2}|\nabla q^*|^2 + \mathbf{e}(q^*\nu) + \int_{I_b\cap\{q^v>q^*\}}((q^v)^2 - (q^*)^2)\sum_j\langle\nu^v_{\xi_j}, \nu^v_{\xi_j}\rangle \geq 0$$

Quantitative estimate of $J_{I_b\cap\{q^v>q^*\}}(v) - J_{I_b\cap\{q^v>q^*\}}(w)$. The minimality of q^* implies

$$\int_{I_b\cap\{q^v>q^*\}} \nabla q^* \cdot \nabla(q^v - q^*) = -\int_{I_b\cap\{q^v>q^*\}} D_q\mathbf{e}(q^*\nu^v)(q^v - q^*)$$

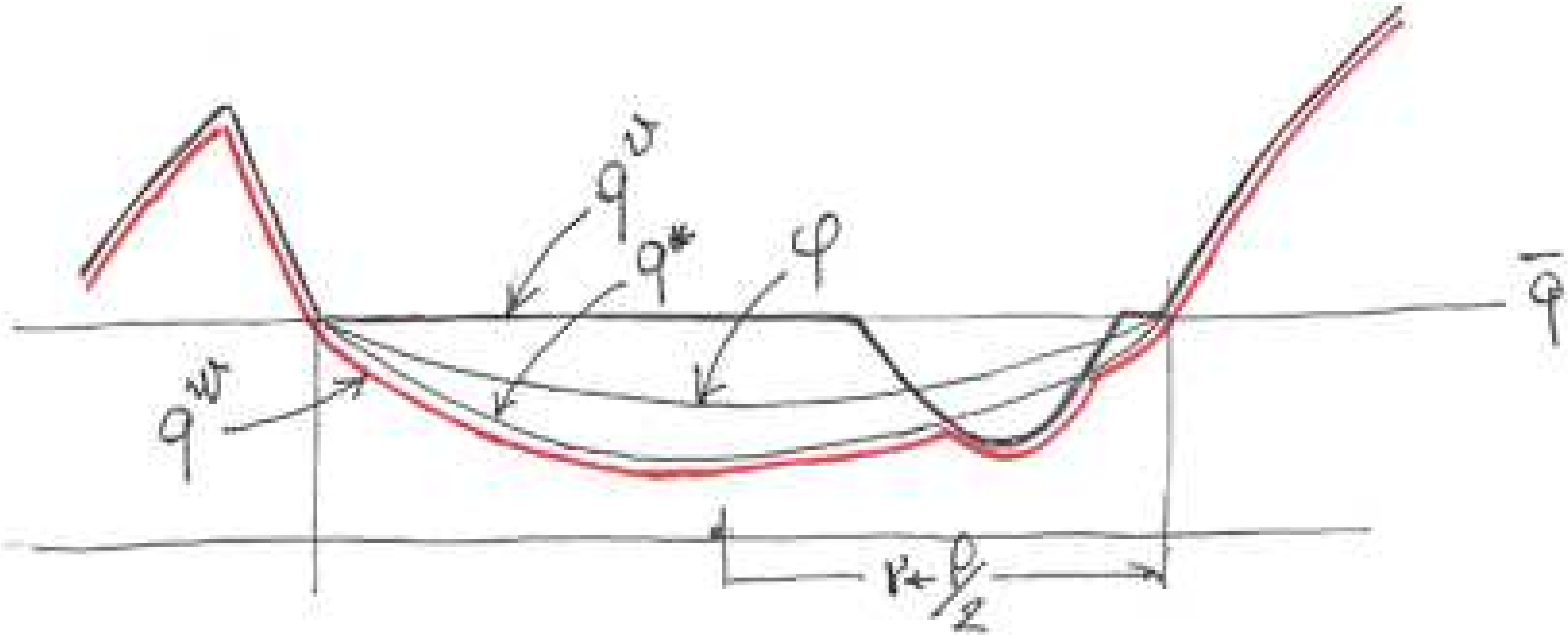
This and the identity

$$\frac{1}{2}(|q^v|^2 - |q^*|^2) = \frac{1}{2}(|q^v - q^*|^2) + D_q\mathbf{e}(q^*\nu^v)(q^v - q^*)$$

imply

$$\begin{aligned} J_{I_b\cap\{q^v>q^*\}}(v) - J_{I_b\cap\{q^v>q^*\}}(w) \\ &\geq J_{I_b\cap\{q^v>q^*\}}\mathbf{e}(q^v\nu) - \mathbf{e}(q^*\nu) - D_q\mathbf{e}(q^*\nu^v)(q^v - q^*) \\ &\geq J_{A_{\bar{q}}\cap B_r}\mathbf{e}(q^v\nu) - \mathbf{e}(q^*\nu) - D_q\mathbf{e}(q^*\nu^v)(q^v - q^*) \\ &\geq J_{A_{\bar{q}}\cap B_r}\mathbf{e}(\bar{q}\nu) - \mathbf{e}(\varphi\nu) - D_q\mathbf{e}(\varphi\nu^v)(\bar{q} - \varphi) \\ &\geq k\mathcal{H}^{n-1}(A_{\bar{q}} \cap B_r). \end{aligned}$$

for some $k > 0$.


q^u
q*
φ
q̄
q^w
r+ℓ/2

4. (Conclusion) Set $\sigma_r := \mathcal{H}^{n-1}(A_{\bar{q}} \cap B_r)$ and note that

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$$\Rightarrow \begin{cases} \sigma_{r+\rho} \geq (1 + \frac{c_1}{C_1})\sigma_r, \\ \sigma_r \leq \frac{C_1}{c_1}(\sigma_{r+\rho} - \sigma_r) \leq C_2(r+\rho)^{n-2}. \end{cases}$$

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Lemma **3** imply that there is $r^\circ > 0$ such that $B_{\xi,r^\circ} \subset \omega$ implies $q^u(\xi) < q^\circ$ therefore we have

$$d(\xi, \partial\omega) \geq r^\circ \Rightarrow q^u(\xi) < q^\circ.$$

The exponential estimate

Assume that $\xi \in \omega$ has $d(xi, \partial\omega) = r^\circ + l$ with $l \geq r^\circ$. Then we have from Lemma 3

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Then, as before we show that $q^u \leq \varphi_l$ where $\varphi_l : B_{\xi,l} \to \mathsf{R}$ is the solution of

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In particular

$$q^u(\xi) \leq \varphi_l(\xi) \leq q^\circ e^{-k_0 l} = q^\circ e^{k_0 r^\circ} e^{-k_0 d(\xi,\partial\omega)} = K_0 e^{-k_0 d(\xi,\partial\omega)}$$