Magnetic Vortices, Nielsen - Olesen - Nambu Strings and Theta Functions

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based on the joint work with S. Gustafson and T. Tzaneteas

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Ginzburg-Landau Equations

Equilibrium states of superconductors (macroscopically) and of the $U(1)$ Higgs model of particle physics are described by the Ginzburg-Landau equations:

$$-\Delta_A \Psi = \kappa^2(1 - |\Psi|^2)\Psi$$
$$\operatorname{curl}^2 A = \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$$

where $(\Psi, A) : \mathbb{R}^d \to \mathbb{C} \times \mathbb{R}^d$, $d = 2, 3$, $\nabla_A = \nabla - iA$, $\Delta_A = \nabla_A^2$, the covariant derivative and covariant Laplacian, respectively, and κ is the Ginzburg-Landau material constant.

Origin of Ginzburg-Landau Equations

Superconductivity. $\Psi : \mathbb{R}^d \to \mathbb{C}$ is called the *order parameter*; $|\Psi|^2$ gives the density of (Cooper pairs of) superconducting electrons. $A : \mathbb{R}^d \to \mathbb{R}^d$ is the magnetic potential. $\mathrm{Im}(\bar{\Psi}\nabla_A\Psi)$ is the superconducting current.

Particle physics. Ψ and A are the Higgs and $U(1)$ gauge (electro-magnetic) fields, respectively. (Part of Weinberg - Salam model of electro-weak interactions/ a standard model.)

Geometrically, A is a connection on the principal $U(1)$- bundle $\mathbb{R}^2 \times U(1)$, and Ψ, a section of this bundle.

Similar equations appear in superfluidity and fractional quantum Hall effect.

Quantization of Flux

From now on we let $d = 2$. Finite energy states (Ψ, A) are classified by the topological degree

$$\deg(\Psi) := \deg\left(\frac{\Psi}{|\Psi|}\bigg|_{|x|=R}\right),$$

where $R \gg 1$. For each such state we have the quantization of magnetic flux:

$$\int_{\mathbb{R}^2} B = 2\pi \deg(\Psi) \in 2\pi\mathbb{Z},$$

where $B := \operatorname{curl} A$ is the magnetic field associated with the vector potential A.

Symmetries

Gauge symmetry: for any sufficiently regular function $\gamma : \mathbb{R}^2 \to \mathbb{R}$,

$$G_\gamma : \ (\Psi(x),\ A(x)) \mapsto (e^{i\gamma(x)}\Psi(x),\ A(x) + \nabla\gamma(x));$$

Translation symmetry: for any $h \in \mathbb{R}^2$,

$$S_h : \ (\Psi(x),\ A(x)) \mapsto (\Psi(x+h),\ A(x+h));$$

Rotation and reflection symmetry: for any $g \in O(2)$

$$R_g : \ (\Psi(x),\ A(x)) \mapsto (\Psi(gx),\ g^{-1}A(gx)).$$

Type I and II Superconductors

Two types of superconductors:

$\kappa < 1/\sqrt{2}$: Type I superconductors, exhibit first-order phase transitions from the non-superconducting state to the superconducting state (essentially, all pure metals);

$\kappa > 1/\sqrt{2}$: Type II superconductors, exhibit second-order phase transitions and the formation of vortex lattices (dirty metals and alloys).

For $\kappa = 1/\sqrt{2}$, Bogomolnyi has shown that the Ginzburg-Landau equations are equivalent to a pair of first-order equations. Using this Taubes described completely solutions of a given degree.

Equivariant Pairs

A key class of solutions is provided by equivariant pairs.

Given a subgroup, G, of the group of rigid motions (a semi-direct product of the groups of translations and rotations), an equivariant pair is a state (Ψ, A) s.t. $\forall g \in G$, $\exists \gamma = \gamma(g)$ s.t.

$$T_g(\Psi, A) = G_\gamma(\Psi, A),$$

where T_g is the action of G and G_γ, of the gauge group.

This leads to two classes of solutions:

$G = O(2) \Longrightarrow$ magnetic vortices (labeled by the equivalence classes of the homomorphisms of S^1 into $U(1)$, i.e. by $n \in \mathbb{Z}$).

$G =$ group of lattice translations $\Longrightarrow$ Abrikosov lattices.

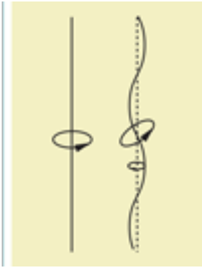
Vortices

"Radially symmetric" (more precisely, *equivariant*) solutions:

$$\Psi^{(n)}(x) = f^{(n)}(r)e^{in\theta} \quad \text{and} \quad A^{(n)}(x) = a^{(n)}(r)\nabla(n\theta),$$

where $n =$ integer and $(r, \theta) =$ polar coordinates of $x \in \mathbb{R}^2$.

$$\deg(\Psi^{(n)}) = n \in \mathbb{Z}. \qquad \text{(Berger-Chen)}$$



$(\Psi^{(n)}, A^{(n)}) =$ the *magnetic n-vortex* (superconductors) or *Nielsen-Olesen* or *Nambu string* (the particle physics).

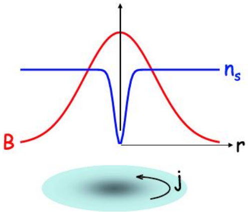
Vortex Profile

The profiles are exponentially localized:

$$|1 - f^{(n)}(r)| \leq ce^{-r/\xi}, \quad |1 - a^{(n)}(r)| \leq ce^{-r/\lambda},$$

Here ξ = *coherence length* and λ = *penetration depth*.

$$\kappa = \lambda/\xi.$$



The exponential decay is due to the Higgs mechanism of mass generation: massless $A \Rightarrow$ massive A, with $m_A = \lambda^{-1}$.

Stability/Instability of Vortices

Theorem

1. *For Type I superconductors all vortices are stable.*
2. *For Type II superconductors, the ± 1-vortices are stable, while the n-vortices with $|n| \geq 2$, are not.*

The statement of Theorem I was conjectured by Jaffe and Taubes on the basis of numerical observations (Jacobs and Rebbi, ...).

Abrikosov Vortex Lattice States

A pair (Ψ, A) for which all the physical characteristics

$$|\Psi|^2, \quad B(x) := \operatorname{curl} A(x), \quad J(x) := \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$$

are doubly periodic with respect to a lattice $\mathcal{L}$ is called the *Abrikosov (vortex) lattice state*.

Theorem. (Ψ, A) is an Abrikosov lattice state if and only if it is an equivariant pair for the group of lattice translations for a lattice $\mathcal{L}$:

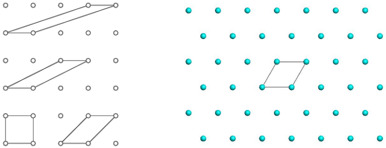
$$S_a(\Psi, A) = G_{\gamma_a}(\Psi, A), \ \forall a \in \mathcal{L}, \tag{1}$$

where $\gamma_a : \mathbb{R}^2 \to \mathbb{R}$ is, in general, a multi-valued differentiable function, with differences of values at the same point $\in 2\pi\mathbb{Z}$.

$$(1) \Rightarrow \gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}.$$

Abrikosov's 'Constant'

By translation, rotation and scaling, $\mathcal{L} \to \mathcal{L}' = r(\mathbb{Z} + \tau\mathbb{Z})$, where $\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$ and $r^2 \operatorname{Im}\tau = 2\pi$



On lattice shapes, τ, define the Abrikosov 'constant':

$$\beta(\tau) = \frac{\langle|\phi|^4\rangle}{\langle|\phi|^2\rangle^2}.$$

Here $\langle\cdot\rangle$ is the average over an elementary cell, Ω', of the lattice $\mathcal{L}'$ and $\phi \neq 0$ is the unique solution of the equation

$$-\Delta_{A_0}\phi = \phi, \qquad A_0 := \frac{1}{2}(-x_2, x_1),$$

satisfying $$\phi(x+\nu) = e^{\frac{i}{2}\nu\wedge x}\phi(x), \qquad \forall \nu \in \mathcal{L}.$$

Existence of Abrikosov Lattices

Let $H_{c2} = \kappa^2$ be the second critical magnetic field, at which the normal material becomes superconducting.
Define the new threshold of the Ginzburg-Landau parameter

$$\kappa_c(\tau) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta(\tau)}\right)} \; (< \tfrac{1}{\sqrt{2}}).$$

Theorem (High magnetic fields)

1) For for every τ and b satisfying $\left|b - \kappa^2\right| \ll 1$ and

- *either $b < \kappa^2$ and $\kappa > \kappa_c(\tau)$ or $b > \kappa^2$ and $\kappa < \kappa_c(\tau)$,*

there exists an Abrikosov lattice solution, with one quantum of flux per cell and with average magnetic flux per cell equal to b.

2) If $\kappa > 1/\sqrt{2}$ (Type II superconductors), then the minimum of the average energy per cell is achieved for the triangular lattice.

Existence of Abrikosov Lattices (Weak MF)

- Similarly, near the first critical magnetic field, H_{c1} (at which the first vortex enters the superconducting sample), we have the following result

Theorem (Low magnetic fields)

For every τ, n and $b > H_{c1}$ (but close to H_{c1}), there exist non-trivial Abrikosov lattice solution, with n quanta of flux per cell and with average magnetic flux per cell $= b$.

References

- Existence for $H \approx H_{c2}$ $(b < H_{c2}$ and $\kappa > \frac{1}{\sqrt{2}} > \kappa_c(\tau))$: Odeh, Barany - Golubitsky - Tursky, Dutour, Tzaneteas - IMS.

Existence for $H \approx H_{c2}$ $(b < \kappa^2$ and $\kappa > \kappa_c(\tau)$ or $b > \kappa^2$ and $\kappa < \kappa_c(\tau))$: Tzaneteas - IMS.

Energy minim. by triangular lattices: Dutour, Tzaneteas - IMS, using results of Aftalion - Blanc - Nier, Nonnenmacher - Voros.

Here the Abrikosov 'constant' $\beta(\tau) = \frac{\langle |\phi|^4 \rangle}{\langle |\phi|^2 \rangle^2}$ plays a cucial role. (ϕ is the unique solution of the equation

$$-\Delta_{A_0}\phi = \phi, \ A_0 := \frac{1}{2}Jx, \quad \phi(x+\nu) = e^{\frac{i}{2}\nu \wedge x}\phi(x), \quad \forall \nu \in \mathcal{L}.)$$

Finite domains: Almog, Aftalion - Serfaty.

- Existence for $H \approx H_{c1}$: Aydi - Sandier and others $(\kappa \to \infty)$ and Tzaneteas - IMS (all κ's).

In the leading approximation the evolution of a superconductor is described by the gradient-flow-type equations

$$\gamma(\partial_t + i\Phi)\Psi = \Delta_A\Psi + \kappa^2(1 - |\Psi|^2)\Psi$$
$$\sigma(\partial_t A - \nabla\Phi) = -\operatorname{curl}^2 A + Im(\bar{\Psi}\nabla_A\Psi),$$

$Re\gamma \geq 0$, the *time-dependent Ginzburg-Landau equations* or the *Gorkov-Eliashberg-Schmidt equations*. (Earlier versions: Bardeen and Stephen and Anderson, Luttinger and Werthamer.)

The last equation comes from two Maxwell equations, with $-\partial_t E$ neglected, (Ampère's and Faraday's laws) and the relations $J = J_s + J_n$, where $J_s = \mathrm{Im}(\overline{\Psi}\nabla_A\Psi)$, and $J_n = \sigma E$.

Time-Dependent Eqns. $U(1)$ Higgs Model

The time-dependent $U(1)$ Higgs model is described by $U(1)-$Higgs (or Maxwell-Higgs) equations ($\Phi = 0$)

$$\partial_t^2 \Psi = \Delta_A \Psi + \kappa^2 (1 - |\Psi|^2)\Psi$$
$$\partial_t^2 A = -\operatorname{curl}^2 A + \operatorname{Im}(\bar{\Psi} \nabla_A \Psi),$$

coupled (covariant) wave equations describing the $U(1)$-gauge Higgs model of elementary particle physics (written here in the *temporal gauge*).

Stability of Abrikosov Lattices. I

Gauge -periodic perturbations: perturbations of the same periodicity as the Abrikosov lattice.

Recall $\kappa_c(\tau) := \frac{1}{\sqrt{2}}\sqrt{1 - \frac{1}{\beta(\tau)}}$. Note that $\kappa_c(\tau) < \frac{1}{\sqrt{2}}$.

Theorem (Tzaneteas - IMS)

The Abrikosov vortex lattice solutions for high magnetic fields are

(i) *asymptotically stable for $\kappa > \kappa_c(\tau)$;*

(ii) *unstable for $\kappa < \kappa_c(\tau)$.*

Stability of Abrikosov Lattices. II

Let $(\Psi_\omega, A_\omega) =$ Abrikosov lattice solution specified by $\omega = (\tau, b)$ and $\mathcal{E}_\Omega(\Psi, A) =$ *Ginzburg-Landau energy functional*

$$\mathcal{E}_\Omega(\Psi, A) := \frac{1}{2}\int_\Omega \left\{ |\nabla_A \Psi|^2 + (\operatorname{curl} A)^2 + \frac{\kappa^2}{2}(|\Psi|^2 - 1)^2 \right\}.$$

Finite-energy perturbations: perturbations satisfying,

$$\lim_{Q \to \mathbb{R}^2} \big(\mathcal{E}_Q(\Psi, A) - \mathcal{E}_Q(\Psi_\omega, A_\omega)\big) < \infty, \text{ for some } \omega.$$

Theorem (Tzaneteas - IMS)

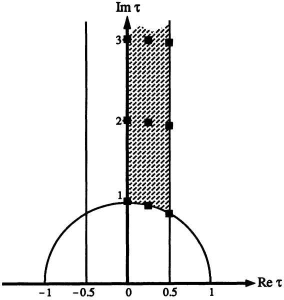
Let $b \approx H_{c2}$ (high magnetic fields).
There is $\gamma(\tau)$ s.t. the Abrikosov vortex lattice solutions are

(i) *asymptotically stable if $\kappa > \frac{1}{\sqrt{2}}$ and $\gamma(\tau) > 0$;*

(ii) *unstable otherwise.*

Gamma Function

The function $\gamma(\tau)$ is invariant under modular group $SL(2,\mathbb{Z})$ and therefore is defined on the Poincaré strip.



Symmetries: $\gamma(-\bar{\tau}) = \gamma(\tau)$ and $\gamma(1-\bar{\tau}) = \gamma(\tau)$
$\Rightarrow$ critical points at $\tau = e^{i\pi/2}$ and $\tau = e^{i\pi/3}$

Work in progress: Estimating $\gamma(\tau)$ and checking the critical points. So far we have $\gamma(e^{i\pi/3}) > 0$

Stability Definition

The stability is defined w.r.to distance to the infinite-dimensional manifold of $\mathcal{L}-$lattice solutions

$$\mathcal{M} = \{T_g^{sym} u_\omega : g \in G\},$$

where $T_g^{sym} = T_\gamma^{gauge} T_h^{trans} T_\rho^{rot}, \ g = (\gamma, h, \rho)$, is the action of the symmetry group

$$G = H^2(\mathbb{R}^2; \mathbb{R}) \times \mathbb{R}^2 \times SO(2)$$

(semi-direct product) on Abrikosov vortex lattices $u_\omega = (\Psi_\omega, A_\omega)$.

Central Step in Proof

Consider the hessian, $\mathcal{E}''(u_\omega)$, of Ginzburg-Landau energy functional $\mathcal{E}(\Psi, A)$ at a Abrikosov lattice solution $u_\omega = (\Psi_\omega, A_\omega)$.

(Recall that the Ginzburg-Landau equations are the Euler-Lagrange equations for $\mathcal{E}$.)

One of the central steps in the proof is to estimate $\mathcal{E}''(u_\omega)$ from below in transversal direction to $\mathcal{M}$.

Magnetic Translations

The key point: $u_\omega = (\Psi_\omega, A_\omega)$ is equivariant $\Longrightarrow$ the Hessian $\mathcal{E}''(u_\omega)$ commutes with magnetic translations,

$$T_s = G_{\gamma_s} S_s,$$

where, recall, S_s is the translation operator $S_s f(x) = f(x+s)$,

$$G_\gamma : \ (\psi(x), \ a(x)) \mapsto (e^{i\gamma(x)}\psi(x), \ a(x) + \nabla\gamma(x));$$

and $\gamma_s : \mathbb{R}^2 \to \mathbb{R}$ is a multi-valued differentiable function, satisfying

$$\gamma_{s+t}(x) - \gamma_s(x+t) - \gamma_t(x) \in 2\pi\mathbb{Z}. \tag{2}$$

$$(2) \quad \Rightarrow \quad T_{s+t} = T_s T_t.$$

($s \to T_s$ is a unitary repres. of $\mathcal{L}$ on $L^2(\mathbb{R}^2; \mathbb{C}) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$.)

Direct Fibre Integral (Bloch Decomposition)

Since the Hessian operator $\mathcal{E}''(u_\omega)$ commutes with T_s, it can be decomposed into the fiber direct integral

$$U\mathcal{E}''(u_\omega)U^{-1} = \int_{\Omega^*}^{\oplus} L_k d\mu_k$$

where Ω^* is the fundamental cell of the reciprocal (dual) lattice, $U : L^2(\mathbb{R}^2; \mathbb{C} \times \mathbb{R}^2) \to \mathscr{H} = \int_{\Omega^*}^{\oplus} \mathscr{H}_k d\mu_k$ is a unitary operator,

$$(Uv)_k(x) = \sum_{s \in \mathcal{L}} e^{-ik \cdot s} T_s v(x)$$

(decomposition into the Bloch waves, $v_k(x) = e^{ik \cdot x}\phi_k(x)$),
$\mathscr{H}_k := \{v \in L^2(\Omega, \mathbb{C} \times \mathbb{R}^2) : T_s v(x) = e^{ik \cdot s} v(x),\ s \in \text{basis}\}$,
L_k is the restriction of the operator $\mathcal{E}''(u_\omega)$ to $\mathscr{H}_k$.

ϑ-function

In the leading order in $\epsilon := \sqrt{\kappa^2 - b}$, the ground state energies of the fiber operators, L_k, are given by

$$\inf L_k = \gamma_k(\tau)\epsilon^2 + O(\epsilon^3),$$

where

$$\gamma_k(\tau) := 2\frac{\langle|\vartheta_k(\tau)|^2|\vartheta_0(\tau)|^2\rangle}{\langle|\vartheta_k(\tau)|^2\rangle\langle|\vartheta_0(\tau)|^2\rangle} + \cdots - \frac{\langle|\vartheta_0(\tau)|^4\rangle}{\langle|\vartheta_0(\tau)|^2\rangle^2}.$$

Here $\vartheta_k(z,\tau)$, $k \in \Omega^*$, are the *modified theta functions*, i.e. entire functions satisfying ($\sqrt{\frac{2\pi}{\operatorname{Im}\tau}}i\frac{1}{2}(a\tau + b) = k_1 + ik_2$)

$$\begin{cases} \vartheta_k(z+1,\tau) = e^{2\pi i a}\vartheta_k(z,\tau), \\ \vartheta_k(z+\tau,\tau) = e^{-2\pi i b}e^{-\pi i\tau z - 2\pi i z}\vartheta_k(z,\tau). \end{cases}$$

Conclusion of Sketch

The relations $\inf L_k = \gamma_k(\tau)\epsilon^2 + O(\epsilon^3)$ and

$$U\mathcal{E}''(u_\omega)U^{-1} = \int_{\Omega^*}^{\oplus} L_k d\mu_k$$

imply

$$\inf \mathcal{E}''(u_\omega) = \underbrace{\inf_{k\in\Omega^*} \gamma_k(\tau)}_{\gamma(\tau)} \epsilon^2 + O(\epsilon^3).$$

Hence the Abrikosov lattice is

- linearly stable if $\gamma(\tau) > 0$
- linearly unstable if $\gamma(\tau) < 0$.

Conclusions

In the context of superconductivity and particle physics, we described

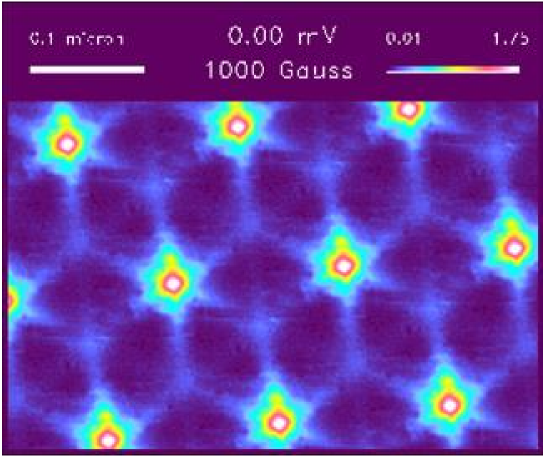
- existence and stability of magnetic vortices and vortex lattices
- a new threshold $\kappa_c(\tau)$ in the Ginzburg-Landau parameter appears in the problem of existence of vortex lattices
- while Abrikosov lattice energetics is governed by Abrikosov function $\beta(\tau)$, a new automorphic function $\gamma(\tau)$ emerges controlling stability of Abrikosov lattices.

We gave some indications how to prove the latter results. While the proof of existence leads to standard theta functions, the proof of stability leads to theta functions with characteristics.

Interesting extensions:

- unconventional/high T_c supercond.,
- Weinberg - Salam model of electro-weak interactions,
- microscopic/quantum theory.

Abrikosov Lattice. Experiment



Thank-you for your attention.

Dynamics of Several Vortices

Consider a dynamical problem with initial conditions, describing several vortices, with the centers at points z_1, $z_2, \ldots$ and with the degrees n_1, n_2, $\ldots$, glued together, e.g.

$$\psi_{\underline{z},\chi}(x) = e^{i\chi(x)} \prod_{j=1}^{m} \psi^{(n_j)}(x - z_j),$$

$$A_{\underline{z},\chi}(x) = \sum_{j=1}^{m} A^{(n_j)}(x - z_j) + \nabla\chi(x) \ ,$$

where $\underline{z} = (z_1, z_2, \ldots)$ and χ is an arbitrary real function.
We will assume that $R(\underline{z}) := \min_{j \neq k} |z_j - z_k| \gg 1$.

Vortex Dynamics: Superconductors

The *superconductor model* (Gustafson - IMS):
For initial data (Ψ_0, A_0) close to some $(\Psi_{\underline{z}_0,\chi_0}, A_{\underline{z}_0,\chi_0})$ with $e^{-R(\underline{z}_0)}/\sqrt{R(\underline{z}_0)} \le \epsilon \ll 1$ we have

$$(\Psi(t), A(t)) = (\Psi_{\underline{z}(t),\chi(t)}, A_{\underline{z}(t),\chi(t)}) + O(\epsilon \log^{1/4}(1/\epsilon))$$

and that the vortex dynamics is governed by the system

$$\gamma_{n_j} \dot{z}_j = -\nabla_{z_j} W(\underline{z}) + O(\epsilon^2 \log^{3/4}(1/\epsilon)).$$

Here $W(\underline{z}) \sim \sum_{j \neq k} (const) n_j n_k \frac{e^{-|z_j - z_k|}}{\sqrt{|z_j - z_k|}}$ is the *effective energy* and $\gamma_n > 0$.

Vortex Dynamics: $U(1)$-Higgs Model

The *Higgs model* (Gustafson - IMS):
For times up to $O\left(\frac{1}{\sqrt{\epsilon}}\log\left(\frac{1}{\epsilon}\right)\right)$, the effective dynamics is given by

$$\gamma_{n_j}\ddot{z}_j = -\nabla_{z_j} W(\underline{z}(t)) + o(\epsilon).$$

with the same effective energy/Hamiltonian

$W(\underline{z}) \sim \sum_{j\neq k}(const)n_j n_k \frac{e^{-|z_j-z_k|}}{\sqrt{|z_j-z_k|}}$ and with $\gamma_n > 0$.

Previous Results

Extensive literature on the *Gross-Pitaevski equation* (superfluids).

The *Gorkov-Eliashberg-Schmidt equations*:
Non-rigorous results: Manton ($\kappa \approx \frac{1}{\sqrt{2}}$), Atiyah - Hitchin ($\kappa \approx \frac{1}{\sqrt{2}}$), Perez - Rubinstein, Chapman-Rubinstein-Schatzman, W.E.

Rigorous results: Stuart, Demoulini - Stuart (both, $\kappa \approx \frac{1}{\sqrt{2}}$), Spirn (independently), Sandier - Serfaty (bounded domains, large κ and h_{ex} below $C\log\kappa$, Tice, Serfaty - Tice (the dynamics with applied field and external current).

No results on the $U(1)$ - Higgs model.