On Singularity Formation Under Mean Curvature Flow

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Mean Curvature Flow

The mean curvature flow is a family of hypersurfaces $M_t \subset \mathbb{R}^{d+1}$ whose smooth immersions $\psi(\cdot, t) : N \to M_t \subset \mathbb{R}^{d+1}$ satisfy the partial differential equation

$$(\partial_t \psi)^N = -H(\psi)$$

where $(\partial_t \psi)^N$ is the normal component of $\partial_t \psi$ and $H(x)$ is the mean curvature of M_t at a point $x \in M_t$.

Applications and Connections

- Material Science (interface motion between different materials or different phases).
- Image recognition.
- Connection to the Ricci flow.
- Topological classification of surfaces and submanifolds.

Some Key Works: Existence

- First mathematical treatment (using geometric measure theory): Brakke [1978];
- Short time existence: Brakke, Huisken, Evans and Spruck, Ilmanen, Ecker and Huisken [1991];
- Weak solutions: Evans and Spruck, Chen, Giga and Goto [1991];

Some Key Works: Singularities

The most interesting problem here is formation of singularities.

- Collapse of convex hypersurfaces: Huisken [1984], extensions: White [2000, 2003], Huisken and Sinestrari [2007-2009];
- Neckpinching for rotationally symmetric hypersurfaces: Grayson, Ecker, Huisken, M. Simon, Dziuk and Kawohl, Smoczyk, Altschuler, Angenent and Giga, Soner and Souganidis [1990-1995];
- MCF with surgery and topological classification of surfaces and submanifolds: Huisken and Sinestrari [2007-2009];
- Nature of the singular set: White [2000, 2003], Colding and Minicozzi [2012].

Explicit Solutions

There are three explicit solutions of MCF:

- Collapsing Euclidean spheres with radii decreasing as $\sqrt{2d(t_* - t)}$;
- Collapsing Euclidean cylinders with radii decreasing as $\sqrt{2(d-1)(t_* - t)}$;

Conjecture [Huisken]: Generic sing. are spheres and cylinders.
Partial results: Huisken, White, Colding and Minicozzi

Results:

- Spherical collapse is asymptotically stable.
- Cylindrical collapse is unstable.
- Cylindrical collapse $\Rightarrow$ neckpinching (stable and universal).

Theorem. (W. Kong-I.M.S.) Let a surface M_0 be close to S^d in H^s, $s > \frac{d}{2} + 1$. Then $\exists t_* < \infty$, s.t. MCF has a solution M_t for $0 \leq t < t_*$ and

- $M_t \to z_*$, for some z_*, as $t \to t_*$;
- M_t are defined by immersions of S^d, satisfying

$$\psi(\omega, t) = z(t) + \rho(\omega, t)\omega,$$

$$\rho(t) = \sqrt{\tau}\left(1 + O_{H^s}(\tau^\beta)\right),$$

with $\tau := 2d(t_* - t)$, $\alpha := \frac{1}{2}(d + \frac{1}{2} - \frac{1}{2d})$ and $\beta := \frac{1}{2}(1 - \frac{1}{2d})$.

Graphs over Cylinders

Our next result deals with initial conditions M_0, which are graphs over $(d+1)-$dimensional cylinders C^{d+1} along the $x_{d+2}-$axis in $\mathbb{R}^{d+2}$,

$$\psi_0(\omega, x) = (u_0(\omega, x)\omega, x).$$

It combines two results, one with Zhou Gang on equivariant graphs (surfaces of revolution), i.e.

$$u_0(\omega, x) \text{ is independent of } \omega,$$

and one in general case with Zhou Gang and Dan Knopf, ArXiv 2012.

Neckpinching

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S) Let $d \geq 1$ and (informally for brevity)

M_0 be a surface close to a cylinder, $\mathcal{C}^{d+1}$,

M_0 has an arbitrary shallow waist and is even w.r.to the waist.

Then M_t is defined by an immersion

$$\psi(\omega, x, t) = (u(\omega, x, t)\omega, x)$$

of C^{d+1}, where $(\omega, x) \in \mathcal{C}^{d+1}$ and $u(\omega, x, t)$ satisfies

(i) There exists a finite time t^* such that $\inf u(\cdot, t) > 0$ for $t < t^*$ and $\lim_{t \to t^*} \inf u(\cdot, t) \to 0$;

(ii) If $u_0 \partial_x^2 u_0 \geq -1$ then there exists a function $u_*(\omega, x) > 0$ such that $u(\omega, x, t) \geq u_*(\omega, x)$ for $\mathbb{R} \backslash \{0\}$ and $t \leq t^*$.

Dynamics of Scaling Parameter

Theorem. (Zhou Gang-S, Zhou Gang-Knopf-S)

(iii) There exist C^1 functions $\zeta(\omega, x, t)$, $\lambda(t)$ and $b(t)$ such that

$$u(\omega, x, t) = \lambda(t)[\sqrt{\frac{d + b(t)y^2}{a(t)}} + \zeta(\omega, y, t)]$$

with $y := x/\lambda(t)$, $a(t) = -\lambda(t)\partial_t\lambda(t)$ and

$$\|\langle y\rangle^{-m}\partial_y^n\zeta(\omega, y, t)\|_\infty \leq cb^2(t), \; m + n = 3.$$

(iv) The parameters $\lambda(t)$ and $b(t)$ satisfy (with $\tau := 2d(t^* - t)$)

$$\lambda(t) = \tau^{\frac{1}{2}}(1 + o(1)) \qquad \text{(scaling eigenvalue)}$$

$$b(t) = -\frac{d}{\ln\tau}(1 + O(\frac{1}{|\ln\tau|^{3/4}})) \qquad \text{(shape eigenvalue).}$$

Comparison with Previous Results

A result similar to (ii) (axi-symmetric surfaces) but for a different set of initial conditions was proven by H.M.Soner and P.E.Souganidis.

The previous result closest to ours is that by S. Angenent and D. Knopf on the axi-symmetric neckpinching for the Ricci flow.

Some ideas of the proof are close to those of Bricmont and Kupiainen on NLH.

All works mentioned above deal with *surfaces of revolution* of barbell shapes (*far from cylinders*) which are either compact (Dirichlet b.c.) or periodic (Neumann b.c.).

These works rely on parabolic maximum principle going back to Hamilton and monotonicity formulae for an entropy functional $\int_{M_t}$ backward heat kernel$(x, t)d\mu_t$, due to Huisken and Giga and Kohn.

Rescaled MCF

We explain what is going on. Rescale the MCF as

$$\varphi(u,\tau) := \lambda^{-1}(t)\psi(u,t), \quad \tau := \int_0^t \frac{dt'}{\lambda(t')^2} .$$

Important point: we do not fix $\lambda(t)$ but consider it as free parameter to be found from MCF. The rescaled MCF satisfies

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \quad a = -\dot{\lambda}\lambda .$$

- The rescaled MCF is a gradient flow for the Huisken functional

$$V_a(\varphi) := \int_{M^\lambda} e^{-\frac{a}{2}|x|^2},$$

where $M^\lambda = \lambda^{-1}(t)M$ is the rescaled surface M.

Self-similar Surfaces

Static solutions of the rescaled MCF

$$(\partial_\tau \varphi)^N = -H(\varphi) + a\langle \varphi, \nu(\varphi)\rangle, \qquad a = -\dot{\lambda}\lambda.$$

- are self-similar surfaces,

$$H(\varphi) - a\langle \nu(\varphi), \varphi\rangle = 0, \qquad a \in \mathbb{R}.$$

We expect that as $\tau \to \infty$, solutions to the rescaled MCF converge to self-similar surfaces.

Hence one wants to classify self-similar surfaces and determine which ones of them are stable.
A key notion here is that of stability $\Rightarrow$ spectrum of the hessian.

Symmetries and Spectrum of Hessian

Theorem. The hessian $\text{Hess}^N V_a(\varphi)$ of $V_a(\varphi)$ in the normal direction at a self-similar $d-$dimensional surface φ has

1. (Colding-Minicozzi) the simple eigenvalue $-2a$,
2. (Colding-Minicozzi) the eigenvalue $-a$ of multiplicity $d+1$,
3. the eigenvalue 0 of multiplicity $\frac{1}{2}(d-1)d$ (unless φ is a sphere).

These eigenvalues are due to *rescaling, translations and rotations* of the surface. The eigenvalue 0 distinguishes between a *sphere, a cylinder and a generic surface*.

Proof. Let $H_a(\varphi) := H(\varphi) - a\varphi \cdot \nu(\varphi)$. To prove say the first statement, we observe that, since $H_{\lambda^{-2}a}(\lambda\varphi) = \lambda^{-1} H_a(\varphi)$,

$$H_{\lambda^{-2}a}(\lambda\varphi) = 0, \ \forall \lambda \in \mathbb{R}_+.$$

Differentiating this equation w.r.to λ at $\lambda = 1$, and reparametrizing the result, we arrive at the desired eigenvalue equation. $\square$

Spectrum and Mean convexity

The spectral information $\Rightarrow$ the geometry of φ.

Theorem. Let φ be a self-similar surface. Then:
(a) (Colding-Minicozzi) For $a > 0$ (shrinker),
$\text{Hess}^N V_a(\varphi) \geq -2a$ iff $H(\varphi) > 0$.

(b) For $a < 0$ (expander), $H(\varphi)$ changes the sign.

Proof. The normal hessian, $\text{Hess}^N V_a(\varphi)$, has a positivity improving property
$\Rightarrow$ the Perron-Frobenius theory applies $\Rightarrow$ the theorem. □

Definition. (Colding-Minicozzi) A self-similar surface is $F-$stable iff the normal hessian satisfies $\text{Hess}^N V_a(\varphi) \geq 0$.

Theorem. (Huisken, Colding-Minicozzi) The only self-similar, $F-$stable surfaces of polyn. growth are planes, spheres and cyl..

For $a = 0$, φ is a minimal surface $\Rightarrow$ cf. Bernstein conjecture.

Linearized Stability

For a self-similar surface φ, consider the manifold of rescaled, translated and rotated self-similar surfaces

$$\mathcal{M}_{\mathrm{self-sim}} := \{ T_g^{\mathrm{rot}} T_z^{\mathrm{transl}} T_\lambda^{\mathrm{scal}} \varphi : (\lambda, z, g) \in \mathbb{R}_+ \times \mathbb{R}^{d+2} \times SO(d+2) \}.$$

Definition (Linearized stability of self-similar surfaces)

We say that a self-similar surface ϕ, with $a > 0$, is *linearly stable* iff

$$\mathrm{Hess}^N V_a(\varphi) > 0 \quad \text{on} \quad \{\text{scaling, transl., rot. modes}\}^\perp.$$

(I.e. the only unstable motions allowed are scaling, transl., rot..)

Spectrum and Stability

The spectral theorem above gives unstable and central manifolds corresponding to the eigenvalues $-2a$, $-a$ and 0.

Hence, if these are the only non-positive eigenvalues, then we expect the stability in the transverse direction to $\mathcal{M}_{\mathrm{self-sim}}$. Otherwise, we expect instability.

Spectral Picture of Collapse: Sphere and Cylinder

For the $d-$sphere of the radius $\sqrt{\frac{a}{d}}$, the normal hessian > 0 on (scaling and translational modes)$^{\perp}$ $\Rightarrow$ by the definition above, it is linearly stable.

For the $(d+1)-$cylinder of the radius $\sqrt{\frac{a}{d}}$, the normal hessian has, in addition to the eigenvalues above,

1. the eigenvalue $-a$ of multiplicity 1, due to translations along the axis of the cylinder,
2. the eigenvalue 0 of multiplicity $d+1$, which originates in a "shape instability".

Hence the $(d+1)-$cylinder is linearly unstable.

Neckpinching

We use the eigenfunctions corresponding to instability eigenvalues,

1. due to translations along the axis of the cylinder (the eigenvalue $-a$ of multiplicity 1),
2. due to a "shape instability" (the eigenvalue 0 of multiplicity $d+1$)

to find the approximate neck profile

$$\varphi(\omega, y, \tau) = (y, \nu_{ab}(y)\omega), \quad \text{where} \quad \nu_{ab} := \sqrt{\frac{d + by^2}{a}}, \ b > 0.$$

Non-positive eigenvalues correspond to central-unstable manifold.
Symmetry eigenvalues correspond to the manifold of shifted and translated cylinders

$$M_{cyl} := \{\lambda(y,\omega) : (\lambda,\ z,\ g) \in G_{sym}\},$$

where $\varphi_{ab}(y,\omega) := (y, \nu_{ab}(y)\omega)$ and
$G_{sym} := \mathbb{R} \times \mathbb{R}^{d+2} \times SO(d+2)$.
The full central-unstable manifold (the manifold of modulated cylinders or necks) is

$$M_{neck} := \{\lambda g \varphi_{ab} : (\lambda,\ z,\ g, a, b) \in \mathcal{P}\},$$

where $\varphi_{ab}(y,\omega) := (y, \nu_{ab}(y)\omega)$ and $\mathcal{P} := G_{sym} \times \mathbb{R}^+ \times \mathbb{R}^+$.

Hessian on the Neck

Consider the Hessian on the neck $\varphi_{ab} = \text{graph}_{\mathcal{C}^{d+1}}\,\nu_{ab}$ in the direction transversal to the neck manifold M_{neck}:

$$Hess^N V_a(\varphi_{ab}) = \underbrace{-\partial_y^2 + ay\partial_y - 2a - \frac{a}{d}\Delta_{\mathbb{S}^d}}_{\text{normal hess on cyl}} + V_{ab}(y,\omega).$$

Now, one can show that

$$Hess^N V_a(\varphi_{ab}) > 0 \quad \text{on} \quad M_{neck}^{\perp}$$

$\Rightarrow$ The evolution is linearly stable in transverse directions.

Orthogonal Decomposition

We return to the original equation and look for surfaces of the form

$$\psi(x, \omega, t) = \lambda(t) g(t) \varphi(y, \omega, \tau) + z(t),$$

where $(\lambda,\ z,\ g) : [0, T) \to \mathbb{R} \times \mathbb{R}^{d+2} \times SO(d+2)$,
$y = \lambda^{-1}(t)x, \quad \tau = \tau(t) := \int_0^t \lambda^{-2}(s) ds$, and
$\varphi(\cdot, \tau) := (y, u(y, \omega, \tau))$, a normal graph over the fixed cylinder.

The time dependent parameters $\lambda(t),\ z(t),\ g(t)$ are chosen so that $\varphi(\cdot, \tau)$ is orthogonal to the non-positive (scaling, translation and rotation) modes of the normal hessian on the cylinder.
Lemma. Given $u(y, \omega, \tau)$, there exist $a(\tau),\ b(\tau)$ s.t.

$$u(y, \omega, \tau) = \nu_{a(\tau), b(\tau)}(y) + f(y, \omega, \tau),$$

with $f(\cdot, \tau) \perp 1,\ a(\tau) y^2 - 1$ in $L^2(\mathbb{R} \times \mathbb{S}^d, e^{-\frac{a(\tau)}{2} y^2} dy d\omega)$.
The vectors 1 and $(1 - ay^2)$ which are almost tangent vectors to the manifold, M_{neck}, provided b is sufficiently small.

Lyapunov-Schmidt Splitting (Effective Equations)

We obtained the splitting

$$u(y,\tau) = V_{a(\tau),b(\tau)}(y) + g(y,\omega,\tau) \tag{1}$$

By the the results above,

$$f(\cdot,\tau) \perp \{\text{symmetry modes}\},\ 1,\ a(\tau)y^2 - 1.$$

Substitute this into (MCF) to obtain

$$\partial_\tau f = -L_{ab} f + F_{ab} + N_{ab}(f) \tag{2}$$

where $L_{ab} := Hess^N V_a(\varphi_{ab})$ is the transverse hessian, $F_{ab} \approx$ a sum of generators of broken symmetries, $N_{ab}(\phi) =$ nonlinearity.

Project (2) on generators of symmetry transformations, $1,\ a(\tau)y^2 - 1 \implies$ the *equations for the parameters* $a,\ b$. *It remains to estimate* f. Here we use $L_{ab} > 0$ on $M^{\perp}_{neck}$.

Key Estimate

Let $U(\tau, \sigma)$, $\tau \geq \sigma \geq 0$, be the propagator generated by $-L_{ab}$. The main step in the proof involves showing the *key propagation estimate*: $\forall g \in X^{\perp}$,

$$\|\langle z\rangle^{-3} U(\tau,\sigma) g\|_{\infty} \lesssim e^{-c(\tau-\sigma)} \|\langle z\rangle^{-3} g\|_{\infty}.$$

Estimating ϕ

Let $U(\tau,\sigma)$ be the propagator generated by $-L_a$. By Duhamel principle we rewrite the differential equation for $\phi(y,\tau)$ as

$$\phi(\tau) = U(\tau,0)\phi(0) + \int_0^\tau U(\tau,\sigma)(F+N)(\sigma)d\sigma.$$

Using the *key propagation estimate* ($\tau \geq \sigma \geq 0$)

$$\|\langle z\rangle^{-3} U(\tau,\sigma) g\|_\infty \lesssim e^{-c(\tau-\sigma)}\|\langle z\rangle^{-3} g\|_\infty,$$

where $g \perp 1,\ a(\tau)y^2 - 1$ in $L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy)$, we obtain

$$M(\tau) \leq M(0) + b^{\frac{1}{2}}(0)P(M(\tau)),$$

where

$$M(\tau) := \max_{\sigma\leq\tau} b^{-\frac{m+n+1}{2}}(\sigma)\|\langle y\rangle^{-m}\partial_y^n \phi(\cdot,\sigma)\|_\infty.$$

Estimating the Linear Propagator. I

Write $L_{ab} = L_{a0} + V$, with $L_{a0} := -\partial_y^2 + ay\partial_y - 2a$ (the normal hessian at the cylinder), and use that V is slowly varying in y to do a multiplicativ perturbation (adiabatic) theory.
For the integral kernel $K(x, y)$ of $U(\tau, \sigma)$ (for simplicity, we do not display the variables of $\mathbb{S}^d$), we have the representation

$$K(x, y) = K_0(x, y)\langle e^V \rangle(x, y),$$

where $K_0(x, y)$ is the integral kernel of the operator $e^{-(\tau-\sigma)L_{a0}}$ and

$$\langle e^V \rangle(x, y) = \int e^{\int_\sigma^\tau V(\omega(s)+\omega_0(s),s)ds} d\mu(\omega).$$

Here $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma, \tau] \to \mathbb{R}$ with the boundary condition $\omega(\sigma) = \omega(\tau) = 0$ and

$$(-\partial_s^2 + a^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x.$$

Estimating the Linear Propagator. II

To estimate $U(x,y)$ for $e^{a(\tau-\sigma)} \le b^{-1/32}(\tau)$ we use the explicit formula

$$K_0(x,y) = 4\pi(1-e^{-2ar})^{-\frac{1}{2}}\sqrt{a}e^{2ar}e^{-a\frac{(x-e^{-ar}y)^2}{2(1-e^{-2ar})}},$$

where $r := \tau - \sigma$, and the bound

$$|\partial_y \langle e^V \rangle(x,y)| \le b^{\frac{1}{2}} r,$$

which follows from the definition of $\langle e^V \rangle$ and the properties

$$V(y,\tau) \ge 0 \text{ and } |\partial_y V(y,\tau)| \lesssim b^{\frac{1}{2}}(\tau).$$

Then we iterate using the semi-group property $\Rightarrow$ estimate of the remainder ϕ.

Extensions

We do not fix the cylinder and look for surfaces of the form

$$\psi(x,\omega,t) = \lambda(t)g(t)\varphi(y,\omega,\tau) + z(t),$$

where $(\lambda,\ z,\ g) : [0,T) \to \mathbb{R} \times \mathbb{R}^{d+2} \times SO(d+2)$,
to be determined later,

$y = \lambda^{-1}(t)x, \qquad \tau = \tau(t) := \int_0^t \lambda^{-2}(s)ds$, and
$\varphi(\cdot,\tau) : \mathcal{C}^{d+1} \to \mathbb{R}^{d+2}$ is a normal graph over the fixed cylinder.

The time dependent parameters $\lambda(t),\ z(t),\ g(t)$ are chosen so that $\varphi(\cdot,\tau)$ is orthogonal to the non-positive (scaling, translation and rotation) modes of the normal hessian on the cylinder.

Then we proceed as before.

Thank-you for your attention.

Comparison with Yang-Mills and Wave Maps Equations

Compare the dynamics for the scaling parameter $\lambda(t)$ for (MCF) and the critical Yang-Mills equation

$$\lambda\ddot{\lambda} = \frac{3}{4}\dot{\lambda}^4,$$

which gives

$$\lambda \approx \sqrt{\frac{2}{3}}\frac{t_* - t}{\sqrt{-\ln(t_* - t)}}.$$

and the critical wave map equation

$$\dot{\lambda}^2 = \lambda\ddot{\lambda}\ln\frac{a}{\lambda\ddot{\lambda}}\ ,\ \ a = 0.122.$$