

Scattering by a semi-infinite lattice of cylinders

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- **Scattering by periodic arrangements of cylinders.**

Elasticity Composite materials.

Water waves Large offshore structures

Electromagnetism Antennas, Photonic crystals, . . .

- **The Helmholtz equation** (Two-dimensional time-harmonic scattering.)

$$\nabla^2 u + k^2 u = 0$$

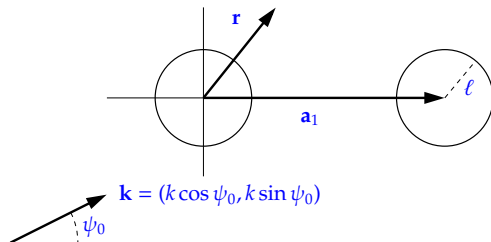
- **Wavefunctions** (separable solutions of the Helmholtz equation)

$$\mathcal{H}_q(\mathbf{r}) = H_q(kr)e^{iq\theta} \quad \mathcal{J}_q(\mathbf{r}) = J_q(kr)e^{iq\theta}$$

- **The multipole expansion method** (linear combination of multipoles).
Representation of the scattered field by one cylinder

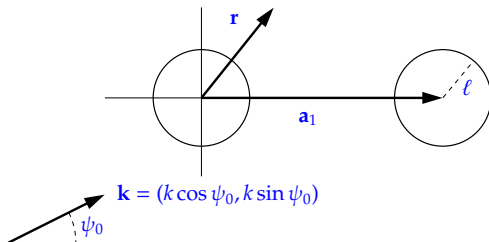
$$u_{sc}(\mathbf{r}) = \sum_{q=-\infty}^{\infty} A^q \mathcal{H}_q(\mathbf{r})$$

Scattering by two cylinders



The total field:
$$u(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + \sum_{q=-\infty}^{\infty} A_0^q \mathcal{H}_q(\mathbf{r}) + \sum_{q=-\infty}^{\infty} A_1^q \mathcal{H}_q(\mathbf{r} - \mathbf{a}_1),$$

Scattering by two cylinders



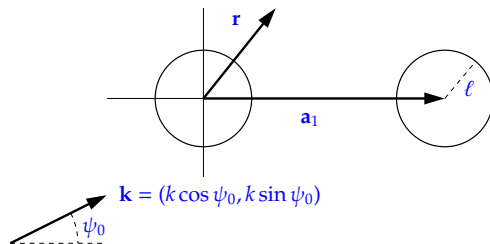
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$$A_n^p + Z_p \sum_{q=-\infty}^{\infty} A_n^q \mathcal{H}_{q-p}(ka_1) = -iZ_p e^{i\mathbf{k}\cdot\mathbf{a}_1} e^{-ip\psi_0}, \quad n = 0, 1, \quad p \in \mathbb{Z}.$$

with

$$Z_p = \begin{cases} J(k\ell)/H(k\ell) & \text{Dirichlet boundary condition} \\ J'(k\ell)/H'(k\ell) & \text{Neumann boundary condition.} \end{cases}$$

Scattering by two cylinders



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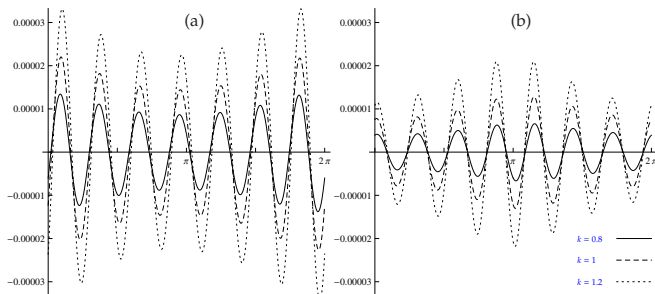
$$A_n^p + Z_p \sum_{q=-Q}^Q A_n^q H_{q-p}(ka_1) = -iZ_p e^{i\mathbf{k}\cdot\mathbf{a}_1} e^{-ip\psi_0}, \quad n = 0, 1, \quad p = -Q, \dots, Q.$$

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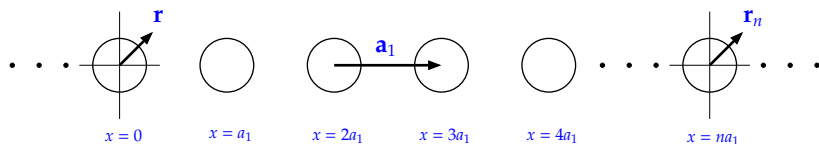
The truncation parameter Q

- *Row*(1953) Verified that for $Q = 6$ the calculations are in excellent agreement with experiments!
- *Linton & Evans* (1990) found that taking $Q = 6$ gave results with accuracy of **four** significant figures.



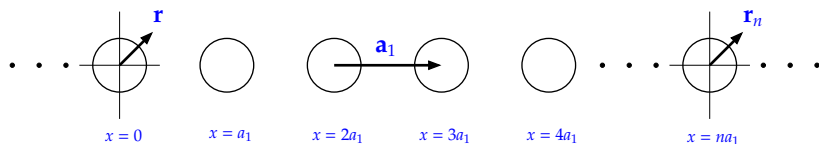
- *Kleinmann & Vainberg*(1994)
For $k\ell \ll 1$, take $Q = 0$ for DBC and $Q = 1$ for NBC with an error of order $(k\ell)^2$

Scattering by an infinite grating of cylinders



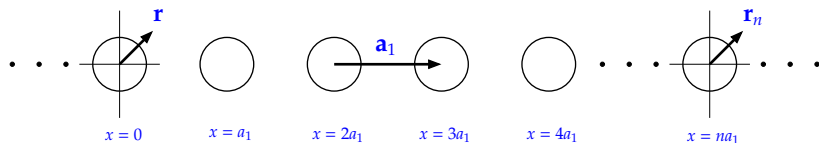
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- The quasi-periodicity condition: $A_n^q = e^{in\mathbf{k}\cdot\mathbf{a}_1} A_0^q, \quad \forall n \in \mathbb{Z}$.

Scattering by an infinite grating of cylinders



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$$A_0^p + Z_p \sum_{q=-Q}^Q A_0^q \sigma_{q-p}^{(1)}(k \cos \psi_0) = -i^p Z_p e^{-ip\psi_0}$$

Lattice Sums

$$\sigma_q^{(1)}(\beta) = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} e^{ij\mathbf{a}_1\beta} \mathcal{H}_q(-j\mathbf{a}_1).$$

The Wiener–Hopf technique

- In 1931, **Norbert Wiener** (1894-1964) and **Eberhard Hopf** (1902-1983), presented a method to solve integral equations of the form

$$\int_0^{\infty} k(x-y)f(y) dy = g(x), \quad 0 < x < \infty.$$

- The **discrete WHT** is used to solve an infinite system of algebraic equations

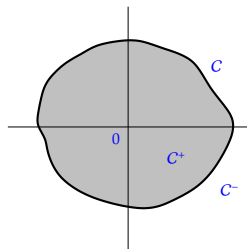
$$\sum_{m=0}^{\infty} K_{n-m}A_m = T_n, \quad n = 0, 1, 2, \dots.$$

$[K_{n-m}]_{n,m=0,1,\dots}$ **Toeplitz matrix**.

The Wiener–Hopf equation

$$K(z)A^+(z) = T^+(z) + T^-(z), \quad z \in C$$

- '+' indicates analyticity in C^+ .
- '-' indicates analyticity in C^- .



Three Fundamental Steps

The factorisation: $K(z) = K^+(z)K^-(z)$, $K^\pm(z)$ are analytic **and zero free** in \mathbb{C}^\pm .

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The sum-split: $\frac{T^+(z)}{K^-(z)} = D^+(z) + D^-(z)$, $D^\pm(z)$ are analytic in C^\pm .

$$K^+(z)A^+(z) - D^+(z) = D^-(z) + \frac{T^-(z)}{K^-(z)} \quad (2)$$

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$J(z)$ is an **entire function** (analytic continuation of both sides)

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$J(z)$ is an **entire function** (analytic continuation of both sides)

Liouville's theorem: If the RHS $\rightarrow 0$ as $z \rightarrow \infty$, then $J(z) \equiv 0$

$$A^+(z) = \frac{D^+(z)}{K^+(z)} \quad \text{and} \quad A_n = \frac{1}{2\pi i} \oint_{z \in C} \frac{D^+(z)}{K^+(z)} z^{-n-1} dz, \quad n = 0, 1, 2,$$

Wiener–Hopf decompositions

- **Rational functions:** trivial decomposition.
- **Meromorphic functions:**
 - Factorisation: Weierstrass product theorem
 - Sum decomposition: Mittag-Leffler theorem

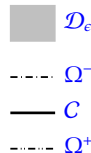
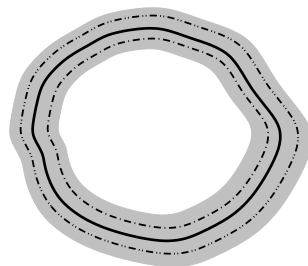
Cauchy's Integral Formula

$$f^\pm(z) = \pm \frac{1}{2\pi i} \oint_{\Omega^\pm} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in \mathbb{C} - \Omega^\pm.$$

$$f(z) = f^+(z) + f^-(z)$$

$$f^\pm(z) = \exp\left(\pm \frac{1}{2\pi i} \oint_{\Omega^\pm} \frac{\log f(\zeta)}{\zeta - z} d\zeta\right),$$

$$f(z) = f^+(z)f^-(z)$$



- **Approximate factorisations:** Padé Approximants.

Remark: WH decomposition are constructed in an ad-hoc manner!

Matrix Wiener–Hopf equation

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$$\mathbf{K}(z)\mathbf{A}^+(z) = \mathbf{T}^+(z) + \mathbf{T}^-(z), \quad z \in \mathbb{C}$$

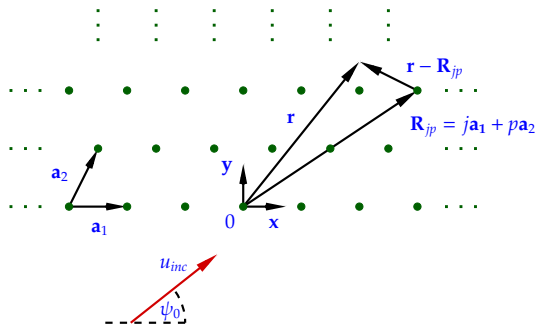
- $\mathbf{K}(z)$ is an $n \times n$ matrix.
- $\mathbf{A}^+(z)$, $\mathbf{T}^-(z)$, and $\mathbf{T}^+(z)$ are $n \times 1$ matrices.

Matrix Wiener-Hopf factorisation (MWHF)

$$\mathbf{K}(z) = \mathbf{K}^+(z)\mathbf{K}^-(z), \quad \det(\mathbf{K}^\pm(z)) \neq 0 \text{ in } \mathbb{C}^\pm.$$

- Gohberg & Krein **proved existence** of MWHF.
- MWHF are **extremely hard!** (even for a rational matrix).
- There are methods to construct a MWHF. However, **no practical applications!**

Scattering by a semi-infinite lattice



$$u_{jp}(\mathbf{r}) = \sum_{q=-Q}^Q A_{jp}^q \mathcal{H}_q(\mathbf{r} - \mathbf{R}_{jp}),$$

$$u_{sc}(\mathbf{r}) = \sum_{j=-\infty}^{\infty} \sum_{p=0}^{\infty} u_{jp}(\mathbf{r}),$$

Quasi-periodicity property

$$A_{jp}^q = e^{ijk \cdot \mathbf{a}_1} A_{0p}^q$$

The infinite system of algebraic equations

$$iZ_\mu^{-1} A_{0n}^\mu + i \sum_{q=-Q}^Q \sum_{p=0}^{\infty} A_{0p}^q S_{n-p}^{q-\mu} = -ie^{ink \cdot \mathbf{a}_2} e^{i\mu(\pi/2 - \psi_0)}, \quad \mu = -Q, \dots, Q, \quad n = 0, 1, \dots,$$

where

$$S_0^q = \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} e^{ijk \cdot \mathbf{a}_1} \mathcal{H}_q(-j\mathbf{a}_1),$$

$$S_n^q = \sum_{j=-\infty}^{\infty} e^{ijk \cdot \mathbf{a}_1} \mathcal{H}_q(n\mathbf{a}_2 - j\mathbf{a}_1), \quad n \in \mathbb{Z} - \{0\}.$$

The scattering angles

$$\cos \psi_j = \cos \psi_0 + \frac{2\pi j}{ka_1}, \quad \text{and} \quad \sin \psi_j = i\gamma(\cos \psi_j)$$

$$\text{where,} \quad \gamma(t) = \begin{cases} \sqrt{t^2 - 1}, & |t| > 1, \\ -i\sqrt{1 - t^2}, & |t| \leq 1. \end{cases}$$

$$\mathcal{M} = \{m \in \mathbb{Z} : |\cos \psi_0 + 2\pi m/ka_1| < 1\},$$

$$\mathcal{N} = \{n \in \mathbb{Z} : |\cos \psi_0 + 2\pi n/ka_1| > 1\},$$

The matrix Wiener–Hopf equation

$$\mathbf{K}(z)\mathbf{A}^+(z) = \mathbf{T}^+(z) + \mathbf{T}^-(z), \quad z \in \mathbb{C}$$

The kernel is the $(2Q + 1) \times (2Q + 1)$ matrix

$$\mathbf{K}(z) = [K_{\mu q}(z)]_{\mu=-Q, \dots, Q; q=-Q, \dots, Q},$$

and the other terms are the column vectors

$$\mathbf{A}^+(z) = [A_q^+(z)]_{q=-Q, \dots, Q},$$

$$\mathbf{T}^+(z) = [T_q^+(z)]_{q=-Q, \dots, Q},$$

$$\text{and } \mathbf{T}^-(z) = [T_q^-(z)]_{q=-Q, \dots, Q}.$$

$$K_{\mu q}(z) = iZ_q^{-1} \delta_{\mu q} + i \sum_{j=-\infty}^{\infty} S_j^{q-\mu} z^j$$

$$T_{\mu}^+(z) = e^{i\mu(\pi/2 - \psi_0)} \frac{ip_0}{z - p_0}.$$

$$A_{0n}^q = \frac{1}{2\pi i} \oint_C A_q^+(z) z^{-n-1} dz,$$

$q = -Q, \dots, Q, \text{ and } n \in \mathbb{Z}.$

The elements of the kernel

$$K_{\mu q}(z) = i(Z_q^{-1} \delta_{\mu q} + S_0^{q-\mu}) + i \sum_{j=1}^{\infty} S_j^{q-\mu} z^j + i \sum_{j=1}^{\infty} S_{-j}^{q-\mu} z^{-j}, \quad (3)$$

$$K_{\mu q}(z) = i(Z_q^{-1} \delta_{\mu q} + S_0^{q-\mu}) - \frac{2i^{\mu-q+1}}{ka_1} \sum_{j=-\infty}^{\infty} \frac{1}{\sin \psi_j} \left(\frac{e^{i(q-\mu)\psi_j} z}{z - p_j} - \frac{e^{i(\mu-q)\psi_j}}{z\tau_j - 1} \right). \quad (4)$$

- $|p_j| = |\tau_j| = 1$ for $j \in \mathcal{M}$, and $p_j^* = \tau_j$ for $j \in \mathcal{N}$
- Also since $iS_n^{q-\mu} = \frac{1}{2\pi i} \oint_{\mathcal{C}} K_{\mu q}(z) z^{-n-1} dz, \quad \forall n \in \mathbb{Z}.$

The location of the poles with respect to \mathcal{C}

$$p_j \in \mathcal{C}^- - \mathcal{C}, \quad \text{and} \quad \tau_j^{-1} \in \mathcal{C}^+ - \mathcal{C}, \quad \forall j \in \mathbb{Z},$$

Symmetry

$$(K_{\mu q}(1/z^*))^* = K_{q\mu}(z) \quad \text{or} \quad (\mathbf{K}(1/z^*))^* = \mathbf{K}(z)$$

$$K_{\mu q}(z) = i(Z_q^{-1} \delta_{\mu q} + S_0^{q-\mu}) + iG_{q-\mu}^+(\mathbf{0}, \beta(z)) + iG_{q-\mu}^-(\mathbf{0}, \beta(z))$$

where

$$G_q^\pm(\mathbf{r}, \beta) = \sum_{m=1}^{\infty} \sum_{j=-\infty}^{\infty} e^{i\mathbf{R}_{j,\mp m} \cdot \beta} \mathcal{H}_q(\mathbf{r} - \mathbf{R}_{j,\mp m}), \quad q \in \mathbb{Z},$$

and

$$\beta(z) = (k \cos \psi_0, \lambda(z)) \quad \text{with} \quad \lambda(z) = \frac{i \log z - \eta_1 k \cos \psi_0}{\eta_2}.$$

Approximation

$$\mathcal{K}_{\mu q}(z) = i(Z_q^{-1} \delta_{\mu q} + S_0^{q-\mu}) - \frac{2j^{\mu-q+1}}{ka_1} \sum_{j=-v}^v \frac{1}{\sin \psi_j} \left(\frac{e^{i(q-\mu)\psi_j} z}{z - p_j} - \frac{e^{i(\mu-q)\psi_j}}{z\tau_j - 1} \right).$$

The approximation parameter $v \geq \max\{|x| : x \in \mathcal{M}\}$

The Wiener–Hopf technique modified

- We sum-split the kernel into plus and minus matrices:

$$\mathbf{K}(z) = \mathbf{K}^+(z) + \mathbf{K}^-(z), \quad \mathbf{K}^\pm(z) = \left[\mathcal{K}_{\mu q}^\pm(z) \right]_{\mu=-Q, \dots, Q; q=-Q, \dots, Q},$$

where

$$\mathcal{K}_{\mu q}^+(z) = i(Z_q \delta_{\mu q} + S_0^{q-\mu}) - \frac{2j^{\mu-q+1}}{ka_1} \sum_{j=-v}^v \frac{e^{i(q-\mu)\psi_j z}}{\sin \psi_j(z - p_j)},$$

$$\mathcal{K}_{\mu q}^-(z) = \frac{2j^{\mu-q+1}}{ka_1} \sum_{j=-v}^v \frac{e^{i(\mu-q)\psi_j}}{\sin \psi_j(z\tau_j - 1)}.$$

- **WH equation:** $\mathbf{K}^+(z)\mathbf{A}^+(z) + \mathbf{K}^-(z)\mathbf{A}^+(z) - \mathbf{T}^+(z) = \mathbf{T}^-(z)$.
- move the poles to the RHS with the aid of

$$\mathbf{M}^-(z) = \left[\frac{2i}{ka_1} \sum_{j=-v}^v \frac{e^{i\mu(\psi_j + \pi/2)}}{\sin \psi_j(z\tau_j - 1)} \sum_{q=-Q}^Q e^{-iq(\psi_j + \pi/2)} \mathbf{A}_q^+(\tau_j^{-1}) \right]_{\mu=-Q, \dots, Q}$$

- **WH equation:** $\mathbf{K}^+(z)\mathbf{A}^+(z) + \mathbf{K}^-(z)\mathbf{A}^+(z) - \mathbf{M}^-(z) - \mathbf{T}^+(z) = \mathbf{T}^-(z) - \mathbf{M}^-(z)$.
- Liouville's theorem dictates:

$$\mathbf{T}^-(z) = \mathbf{M}^-(z)$$

Summary of what we know so far

- The elements of $\mathbf{T}^-(z)$ are given by

$$T_{\mu}^-(z) = \frac{2i}{ka_1} \sum_{j=-v}^v \frac{e^{i\mu(\psi_j+\pi/2)}}{\sin \psi_j(z\tau_j - 1)} X_j, \quad \mu = -Q, \dots, Q,$$

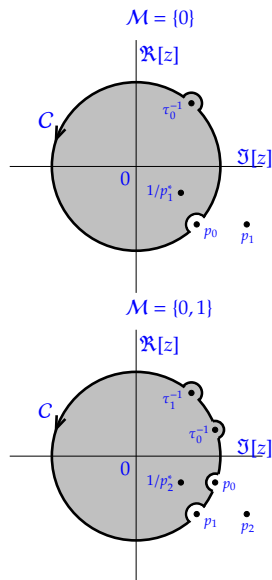
where

$$X_j = \sum_{q=-Q}^Q e^{-iq(\psi_j+\pi/2)} A_q^+(\tau_j^{-1}), \quad j = -v, \dots, v,$$

- The elements of $\mathbf{A}^+(z)$ are rational functions analytic inside the unit circle.

The objective

Form a system of $2v + 1$ algebraic equations to determine X_j .



The kernel in terms of a polynomial matrix

$$\mathbf{K}(z) = \frac{1}{\mathcal{P}(z)\mathcal{T}(z)}\mathbf{P}(z),$$

where $\mathbf{P}(z)$ is a polynomial matrix and

$$\mathcal{P}(z) = \prod_{n=-\nu}^{\nu} (z - p_n), \quad \mathcal{T}(z) = \prod_{n=-\nu}^{\nu} (z - \tau_n^{-1}),$$

$D(z)$ denotes the determinant of $\mathbf{P}(z)$

- $\deg(D(z)) = 2(2Q + 1)(2\nu + 1)$,
- $D(z) = 0 \Leftrightarrow D(1/z^*) = 0$,
- $D(z) = \mathcal{P}^{2Q}(z)\mathcal{T}^{2Q}(z)D_1(z)$,

where $D_1(z)$ is a polynomial of degree $4\nu + 2$.

The adjugate of $\mathbf{P}(z)$

$$\text{adj}(\mathbf{P}(z)) = \mathcal{P}^{2Q-1}(z)\mathcal{T}^{2Q-1}(z)\mathbf{F}(z),$$

for some polynomial matrix $\mathbf{F}(z)$.

The solution

- The WH equation:

$$\frac{1}{\mathcal{P}(z)\mathcal{T}(z)}\mathbf{P}(z)\mathbf{A}^+(z) = \mathbf{T}^+(z) + \mathbf{T}^-(z).$$

- Multiply with the adjugate of $\mathbf{P}(z)$ to obtain

$$D_1(z)\mathbf{A}^+(z) = \mathbf{F}(z)(\mathbf{T}^+(z) + \mathbf{T}^-(z)). \quad (1)$$

- $D_1(z_m) = 0 \Leftrightarrow D_1(1/z_m^*) = 0$ for $m = -v, \dots, v$. Therefore if $|z_m| > 1$,

The system of equations for X_j

$$\mathbf{F}(1/z_m^*)(\mathbf{T}^+(1/z_m^*) + \mathbf{T}^-(1/z_m^*)) = 0.$$

- Once we determine X_j , then $\mathbf{A}^+(z)$ is known and

$$A_{0n}^\mu = \frac{1}{2\pi i} \oint_C A_\mu^+(z) z^{-n-1} dz, \quad \mu = -Q, \dots, Q, \quad n = 0, 1, \dots,$$

$$A_{0n}^\mu = - \sum_{m=-v}^v z_m^{-n-1} \rho_m^\mu, \quad \mu = -Q, \dots, Q, \quad n = 0, 1, \dots,$$

with

$$\rho_m^\mu = \operatorname{Res}_{z=z_m} A_\mu^+(z), \quad \mu = -Q, \dots, Q.$$

The far field pattern

The scattered field in the lower half plane

$$u_{sc}(\mathbf{r}) \sim \sum_{j \in \mathcal{M}} \frac{2X_j}{ka_1 \sin \psi_j} e^{ik(x \cos \psi_j + y \sin \psi_j)} \quad \text{as } y \rightarrow -\infty,$$

The scattered field in the upper half plane

- If the determinant of the kernel is zero free on the unit circle, then

$$u_{sc}(\mathbf{r}) \sim -e^{ik(x \cos \psi_0 + y \sin \psi_0)} \quad \text{as } y \rightarrow +\infty,$$

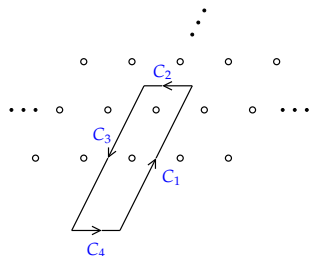
- If the determinant of the kernel has a pair of zeros z_0 and z'_0 on the unit circle, and z_0 corresponds to a pole of $\mathbf{A}^+(z)$, then

$$u_{sc}(\mathbf{r}) \sim -\frac{1}{z_0} \sum_{q=-Q}^Q \rho_0^q G_q^{(2)}(\mathbf{r}, \beta(z_0)) - e^{ik(x \cos \psi_0 + y \sin \psi_0)} \quad \text{as } y \rightarrow +\infty,$$

where

$$G_q^{(2)}(\mathbf{r}, \beta) = \sum_{m=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} e^{i\mathbf{R}_{jm} \cdot \beta} \mathcal{H}_q(\mathbf{r} - \mathbf{R}_{jm}), \quad q \in \mathbb{Z}.$$

Conservation of energy



- Let C be a parallelogram with vertices

$$\pm \frac{1}{2} \mathbf{a}_1 \pm \left(N + \frac{1}{2}\right) \mathbf{a}_2.$$

- $$I_j = -\frac{P_0 \omega}{2} \oint_{C_j} u(\mathbf{r}) \left(\frac{\partial u(\mathbf{r})}{\partial n} \right)^* ds,$$

The principal of conservation of energy:

$$I_1 + I_2 + I_3 + I_4 = 0 \Rightarrow I_2 + I_4 = 0.$$

Calculations indicate that this is satisfied up to 6 significant figures.

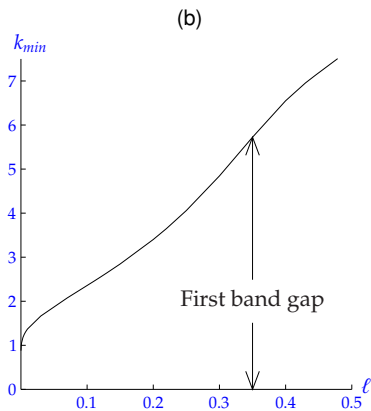
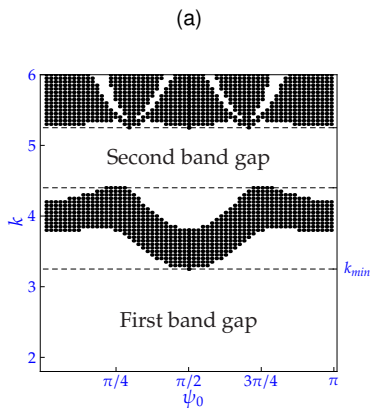
Using I_2 to determine which zero on the unit circle corresponds to a pole of $\mathbf{A}^+(z)$

Given a pair of zeros z_0 and z'_0 on the unit circle

- $I_2(z'_0) < 0$ means that energy is propagating \downarrow across C_2 . (Unphysical)
 $\mathbf{A}^+(z)$ must be analytic at this point.
- $I_2(z_0) > 0$ means that energy is propagating \uparrow across C_2 . (Right!)
 $\mathbf{A}^+(z)$ has a pole at this point.

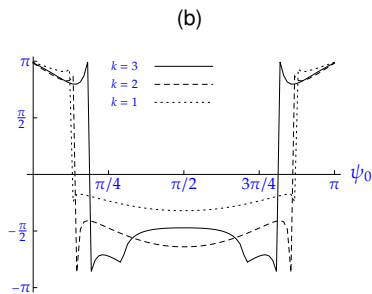
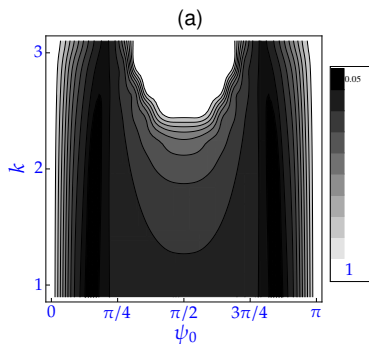
The band structure of square lattice (sound soft cylinders)

(a) $\ell = 0.187$



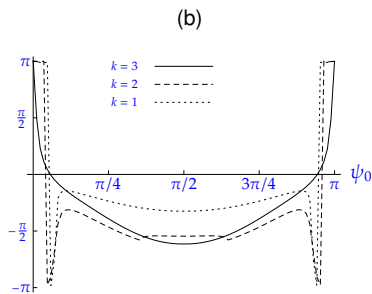
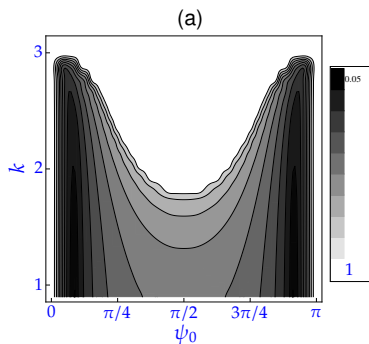
The band structure of square lattice (sound hard cylinders)

$\ell = 0.25$



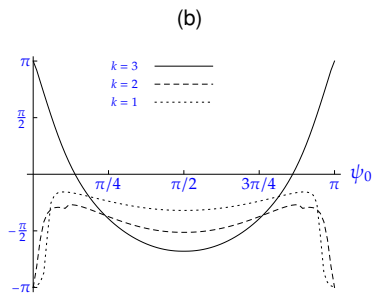
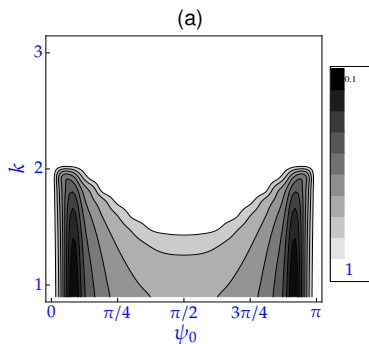
The band structure of square lattice (sound hard cylinders)

$$\ell = 0.42$$

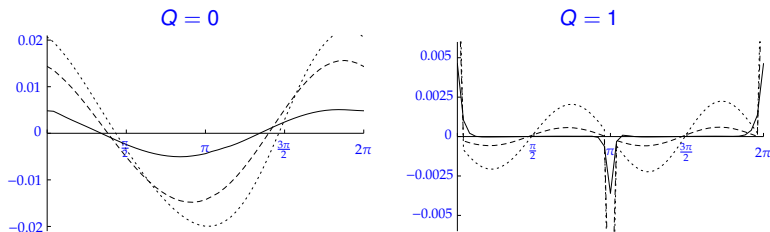


The band structure of square lattice (sound hard cylinders)

$$\ell = 0.472$$



- The method is efficient at **low frequencies**.
 $\ell = 0.01$ (solid line), $\ell = 0.05$ (dashed line) and $\ell = 0.1$ (dotted line)



- **Higher frequencies** Q and ν need to be sufficient large
 - Increasing Q causes the dimension of the matrix kernel to increase.
 - Increasing ν introduces new zeros. ($|z_\nu| \gg 1$ and $|1/z_\nu| \ll 1$).
 - More than one pair of zeros on the unit circle introduces greater complexity.

Thank you!