

Relative entropy methods for relaxation limits

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Main motivation for relaxation limits

Hydrodynamic limit for the Boltzmann equation:

$$\nu f_t + \xi \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f) \quad (1)$$

ν : Mach number and ε : Knudsen number

if $\nu = \varepsilon$

(1) \longrightarrow Navier–Stokes equations as $\varepsilon \downarrow 0$

if $\nu = 1$

(1) \longrightarrow Euler equations as $\varepsilon \downarrow 0$

Simplest discrete velocity model (diffusive relaxation): Carleman's equations

$$\begin{cases} \partial_t f_1 + \frac{1}{\varepsilon} \partial_x f_1 = \frac{1}{\varepsilon^2} (f_2^2 - f_1^2) \\ \partial_t f_2 - \frac{1}{\varepsilon} \partial_x f_2 = \frac{1}{\varepsilon^2} (f_1^2 - f_2^2) \end{cases}$$

$\rho = f_1 + f_2$ as $\varepsilon \downarrow 0$ satisfies $\rho_t = \frac{1}{2}(\log(\rho))_{xx}$

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Relative entropy³

U (weak, entropy) solution and \bar{U} *smooth solution* of systems of conservation (or balance) laws and (η, q_α) a convex entropy–entropy flux pair

Compute $\eta(U|\bar{U})_t + \sum_\alpha \partial_\alpha q_\alpha(U|\bar{U})$ for

$$\eta(U|\bar{U}) = \eta(U) - \eta(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (U - \bar{U})$$

$$q_\alpha(U|\bar{U}) = q_\alpha(U) - q_\alpha(\bar{U}) - \nabla_U \eta(\bar{U}) \cdot (F_\alpha(U) - F_\alpha(\bar{U}))$$

This shall lead to a *stability estimate*, used in many different contexts. Recently:

- ▶ Hyperbolic relaxation: L., Tzavaras *ARMA* '06; Tzavaras *Commun. Math. Sci.* '05; Berthelin, Vasseur *SIMA* '05; Berthelin, Tzavaras, Vasseur *J. Stat. Physics* '09;
- ▶ Weak–strong uniqueness: Demoulini, Stuart, Tzavaras *ARMA*; Feireisl, Novotny *ARMA* '12

³Dafermos, *ARMA* '79; DiPerna, *Indiana U. Math. J.* '79

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Relative entropy for stress relaxation⁴Hyperbolic–hyperbolic relaxation limit $\varepsilon \downarrow 0$ for

$$\begin{aligned}\partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\varepsilon} (S_{i\alpha} - T_{i\alpha}(F))\end{aligned}\tag{2}$$

In (2), $i, \alpha = 1, 2, 3$, F is the deformation gradient, v is the velocity, S is the stress, decomposed as follows:

$$S = f(F) + \int_{-\infty}^t \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(t-\tau)} h(F(\cdot, \tau)) d\tau$$

Its formal limit is the elasticity system

$$\begin{aligned}\partial_t \bar{F}_{i\alpha} &= \partial_\alpha \bar{v}_i \\ \partial_t \bar{v}_i &= \partial_\alpha T_{i\alpha}(\bar{F})\end{aligned}\tag{3}$$

⁴L., Tzavaras ARMA '06

Dissipation of mechanical energy

Possible framework:

$$\begin{aligned} T(F) &= \nabla_F W(F) = f(F) + h(F), \\ f(F) &= \nabla_F W_I(F), \quad h(F) = -\nabla_F W_V(F) \end{aligned} \quad (a)$$

Dissipation of the mechanical energy:

$$\begin{aligned} \partial_t \left(\frac{1}{2} |v|^2 + \Psi(F, S - f(F)) \right) - \partial_\alpha (v_i S_{i\alpha}) \\ + \frac{1}{\varepsilon} (F_{i\alpha} - h_{i\alpha}^{-1}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) = 0 \end{aligned} \quad (4)$$

with $\Psi(F, A) = W_I(F) + A \cdot F + G(A)$ and G is a convex function such that $\nabla_A G = -h^{-1}$

$$\begin{aligned} (F - h^{-1}(S - f(F))) \cdot (S - T(F)) \\ = (\nabla_A G(A) - \nabla_A G(h(F))) \cdot (A - h(F)) \geq 0 \end{aligned}$$

Relative entropy

Relative entropy $\mathcal{E}_r(v, F, S \mid \bar{v}, \bar{F}, h(\bar{F}))$ (with the normalization $\Psi(F, h(F)) = W(F)$):

$$\begin{aligned} \mathcal{E}_r := & \frac{1}{2} |v - \bar{v}|^2 + \Psi(F, S - f(F)) - \Psi(\bar{F}, h(\bar{F})) \\ & - \nabla_F \Psi(\bar{F}, h(\bar{F})) \cdot (F - \bar{F}) - \nabla_A \Psi(\bar{F}, h(\bar{F})) \cdot (S - f(F) - h(\bar{F})) \end{aligned}$$

Relative entropy identity:

$$\begin{aligned} & \partial_t \mathcal{E}_r - \partial_\alpha \left((v_i - \bar{v}_i)(S_{i\alpha} - T_{i\alpha}(\bar{F})) \right) \\ & + \frac{1}{\varepsilon} (F_{i\alpha} - h_{i\alpha}^{-1}(S - f(F)))(S_{i\alpha} - T_{i\alpha}(F)) \\ & = (\partial_\alpha \bar{v}_i) \left(T_{i\alpha}(F) - T_{i\alpha}(\bar{F}) - \frac{\partial^2 W}{\partial F_{i\alpha} \partial F_{j\beta}}(\bar{F})(F_{j\beta} - \bar{F}_{j\beta}) \right) \\ & + (\partial_\alpha \bar{v}_i)(S_{i\alpha} - T_{i\alpha}(F)) \end{aligned} \tag{5}$$

Stability estimate/1

Framework: There exists constants $\gamma_I > \gamma_V > 0$ and $M > 0$ such that

$$\nabla_F^2 W_I(F) \geq \gamma_I I > \gamma_V I \geq \nabla_F^2 (W_I - W)(F) > 0, \quad (\text{b})$$

$$|\nabla_F^2 W_I(F)| \leq M, \quad |\nabla^3 W(F)| \leq M, \quad \forall F \quad (\text{c})$$

Then

$$\mathcal{E}_r \geq c(|v - \bar{v}|^2 + |F - \bar{F}|^2 + |A - h(\bar{F})|^2)$$

for a positive $c > 0$. Condition (b) is roughly equivalent to what is called sub-characteristic condition in the theory of relaxation

$$\begin{aligned} & (F - h^{-1}(S - f(F))) \cdot (S - T(F)) \\ &= (\nabla_A G(A) - \nabla_A G(h(F))) \cdot (A - h(F)) \\ &\geq \frac{1}{\gamma_V} |A - h(F)|^2 = \frac{1}{\gamma_V} |S - T(F)|^2 \end{aligned}$$

Stability estimate/2

Theorem

Let $(v^\varepsilon, F^\varepsilon, S^\varepsilon)$ smooth solutions of (2) and (\bar{v}, \bar{F}) smooth solutions of (3) defined on $\mathbb{R}^3 \times [0, T]$. Then, under (a), (b), (c), the relative energy \mathcal{E}_r satisfies (5), and, for $R > 0$, there exist constants s and $C > 0$ independent of ε such that

$$\int_{|x| < R} \mathcal{E}_r(x, t) dx \leq C \left(\int_{|x| < R+st} \mathcal{E}_r(x, 0) dx + \varepsilon \right)$$

If the data satisfy $\int_{|x| < R+st} \mathcal{E}_r(x, 0) dx \rightarrow 0$ as $\varepsilon \downarrow 0$, then

$$\sup_{t \in [0, T]} \int_{|x| < R} (|v^\varepsilon - \bar{v}|^2 + |F^\varepsilon - \bar{F}|^2 + |A^\varepsilon - h(\bar{F})|^2) dx \rightarrow 0$$

as $\varepsilon \downarrow 0$

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Relative entropy and *diffusive* relaxation

When dealing with a *diffusive relaxation*, we must compare (weak, entropy) solutions U of hyperbolic systems of balance laws (with a damping diffusive relaxation term) and *smooth solutions* \bar{U} of its parabolic limit. This will lead to two main differences w.r.t. the standard theory:

- ▶ the limit \bar{U} is an ε -dependent solution that adapts itself in the relaxation process $\varepsilon \downarrow 0$;
- ▶ both solutions U and \bar{U} are energy dissipative, while in the usual theory and energy dissipative (weak) solution is compared to an energy conservative (smooth) solution.

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The model

Isentropic gas dynamics in three space dimensions with a damping term:

$$\begin{cases} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m, \end{cases} \quad (6)$$

$t \in \mathbb{R}$, $x \in \mathbb{R}^3$, density $\rho \geq 0$ and momentum flux $m \in \mathbb{R}^3$. The pressure $p(\rho)$ satisfies $p'(\rho) > 0$ which makes the system hyperbolic. γ -law: $p(\rho) = k\rho^\gamma$ with $\gamma \geq 1$ and $k > 0$. In the diffusive relaxation limit $\varepsilon \downarrow 0$, solutions of (6) formally converge to those of the porous medium equation

$$\bar{\rho}_t - \Delta_x p(\bar{\rho}) = 0$$

Hilbert's expansion/1

We now use the standard Hilbert's expansion

$$\begin{aligned}\rho &= \rho_0 + \varepsilon \rho_1 + \varepsilon^2 \rho_2 + \dots, \\ m &= m_0 + \varepsilon m_1 + \varepsilon^2 m_2 + \dots,\end{aligned}$$

into (6) and into

$$\eta(\rho, m)_t + \frac{1}{\varepsilon} \operatorname{div}_x q(\rho, m) = -\frac{1}{\varepsilon^2} \nabla_m \eta(\rho, m) \cdot m = -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} \leq 0$$

for the mechanical energy $\eta(\rho, m) = \frac{1}{2} \frac{|m|^2}{\rho} + h(\rho)$

and the associated flux $q(\rho, m) = \frac{1}{2} m \frac{|m|^2}{\rho^2} + m h'(\rho)$

$$h''(\rho) = \frac{p'(\rho)}{\rho}; \quad \rho h'(\rho) = p(\rho) + h(\rho)$$

Hilbert's expansion/2

From the equations we get:

$$O(\varepsilon^{-1}) \operatorname{div}_x m_0 = 0$$

$$O(1) \partial_t \rho_0 + \operatorname{div}_x m_1 = 0$$

$$O(\varepsilon) \partial_t \rho_1 + \operatorname{div}_x m_2 = 0$$

$$O(\varepsilon^{-2}) m_0 = 0$$

$$O(\varepsilon^{-1}) - m_1 = \nabla_x p(\rho_0)$$

$$O(1) - m_2 = \nabla_x (p'(\rho_0) \rho_1)$$

$$O(\varepsilon) - m_3 = \partial_t m_1 + \operatorname{div}_x \left(\frac{m_1 \otimes m_1}{\rho_0} \right) + \nabla_x \left(p'(\rho_0) \rho_2 + \frac{1}{2} p''(\rho_0) \rho_1^2 \right)$$

In particular, we recover the equilibrium relation $m_0 = 0$ for the state variables, the Darcy's law $m_1 = -\nabla_x p(\rho_0)$, and that ρ_0 satisfies porous medium equation

Hilbert's expansion/3

From the entropy we get:

$$O(1) h(\rho_0)_t + \operatorname{div}_x (m_1 h'(\rho_0)) = -\frac{|m_1|^2}{\rho_0}$$

$$O(\varepsilon) \partial_t (h'(\rho_0)\rho_1) + \operatorname{div}_x (m_2 h'(\rho_0) + m_1 h''(\rho_0)\rho_1)$$

$$= |m_1|^2 \frac{\rho_1}{\rho_0^2} - 2 \frac{m_1 \cdot m_2}{\rho_0}$$

Thus we recover the entropy dissipation relation associated to the porous medium equation for ρ_0

$$h(\rho)_t - \operatorname{div}_x (h'(\rho)\nabla_x \rho) = -\frac{|\nabla_x \rho|^2}{\rho}$$

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Reformulation of the limiting equation

We rewrite $\bar{\rho}_t - \Delta_x p(\bar{\rho}) = 0$ as follows

$$\begin{aligned} \bar{\rho}_t + \frac{1}{\varepsilon} \partial_{x_i} \bar{m}_i &= 0 \\ \bar{m}_t + \frac{1}{\varepsilon} \partial_{x_i} f_i(\bar{\rho}, \bar{m}) &= -\frac{1}{\varepsilon^2} \bar{m} + e(\bar{\rho}, \bar{m}) \end{aligned} \quad (7)$$

for $(\bar{\rho}, \bar{m} = -\varepsilon \nabla_x p(\bar{\rho}))$ and the *error term*

$$\begin{aligned} \bar{e} := e(\bar{\rho}, \bar{m}) &= \frac{1}{\varepsilon} \operatorname{div}_x \left(\frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} \right) - \varepsilon \partial_t \nabla_x p(\bar{\rho}) \\ &= \varepsilon \operatorname{div}_x \left(\frac{\nabla_x p(\bar{\rho}) \otimes \nabla_x p(\bar{\rho})}{\bar{\rho}} \right) - \varepsilon \nabla_x (p'(\bar{\rho}) \Delta_x p(\bar{\rho})) \\ &= O(\varepsilon) \end{aligned}$$

Relative entropy/1

The relative entropy is of the form

$$\begin{aligned} \eta(\rho, m | \bar{\rho}, \bar{m}) &:= \eta(\rho, m) - \eta(\bar{\rho}, \bar{m}) - \eta_\rho(\bar{\rho}, \bar{m})(\rho - \bar{\rho}) \\ &\quad - \nabla_m \eta(\bar{\rho}, \bar{m}) \cdot (m - \bar{m}) \\ &= \frac{1}{2} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 + h(\rho | \bar{\rho}) \end{aligned}$$

while the corresponding relative entropy-flux reads

$$\begin{aligned} q_i(\rho, m | \bar{\rho}, \bar{m}) &:= q_i(\rho, m) - q_i(\bar{\rho}, \bar{m}) - \eta_\rho(\bar{\rho}, \bar{m})(m_i - \bar{m}_i) \\ &\quad - \nabla_m \eta(\bar{\rho}, \bar{m}) \cdot (f_i(\rho, m) - f_i(\bar{\rho}, \bar{m})) \\ &= \frac{1}{2} m_i \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 + \rho (h'(\rho) - h'(\bar{\rho})) \left(\frac{m_i}{\rho} - \frac{\bar{m}_i}{\bar{\rho}} \right) + \frac{\bar{m}_i}{\bar{\rho}} h(\rho | \bar{\rho}) \end{aligned}$$

Relative entropy/2

Proposition

Let (ρ, m) be a weak entropy solution of (6) and let $(\bar{\rho}, \bar{m})$ be a smooth solution of (7). Then,

$$\eta(\rho, m | \bar{\rho}, \bar{m})_t + \frac{1}{\varepsilon} \operatorname{div}_x q(\rho, m | \bar{\rho}, \bar{m}) \leq -\frac{1}{\varepsilon^2} R(\rho, m | \bar{\rho}, \bar{m}) - Q - E,$$

where

$$\begin{aligned} Q &= \frac{1}{\varepsilon} \nabla_{(\rho, m)}^2 \eta(\bar{\rho}, \bar{m}) \begin{pmatrix} \bar{\rho}_{x_i} \\ \bar{m}_{x_i} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ f_i(\rho, m | \bar{\rho}, \bar{m}) \end{pmatrix} \\ &= - \sum_{i,j} \partial_{x_i x_j} h'(\bar{\rho}) f_{ij}(\rho, m | \bar{\rho}, \bar{m}) \end{aligned}$$

$$R(\rho, m | \bar{\rho}, \bar{m}) = \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 \quad E = e(\bar{\rho}, \bar{m}) \cdot \frac{\rho}{\bar{\rho}} \left(\frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right)$$

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Control of the quadratic term Q

Lemma

If $p(\rho)$ satisfies $p''(\rho) \leq A \frac{p'(\rho)}{\rho} \quad \forall \rho > 0$ for some $A > 0$, then $h(\rho)$ verifies

$$p(\rho | \bar{\rho}) \leq ch(\rho | \bar{\rho}) \quad \forall \rho, \bar{\rho} > 0$$

for a given constant $c > 0$. Moreover, there exists a $C > 0$ such that for any fixed i

$$|f_i(\rho, m | \bar{\rho}, \bar{m})| \leq C\eta(\rho, m | \bar{\rho}, \bar{m})$$

$$f_{ij}(\rho, m | \bar{\rho}, \bar{m}) = \rho \left(\frac{m_i}{\rho} - \frac{\bar{m}_i}{\bar{\rho}} \right) \left(\frac{m_j}{\rho} - \frac{\bar{m}_j}{\bar{\rho}} \right) + p(\rho | \bar{\rho}) \delta_{ij}$$

Remark: For a γ -law gases:

$$\gamma > 1, h(\rho) = \frac{1}{\gamma-1} p(\rho); \quad \gamma = 1, p(\rho | \bar{\rho}) = 0$$

Possible framework

We assume

(H₁) $\bar{\rho}$ is a smooth, positive (C^3) periodic solution of the multidimensional porous media equation

$(\rho \geq 0, m)$ is a (periodic) *dissipative weak solution* of (6):

$$\iint_{[0,+\infty) \times \mathbb{T}^3} \left[\left(\frac{1}{2} \frac{|m|^2}{\rho} + h(\rho) \right) \dot{\theta}(t) - \frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} \theta(t) \right] dx dt \\ + \int_{\mathbb{T}^3} \left(\frac{1}{2} \frac{|m|^2}{\rho} + h(\rho) \right) \Big|_{t=0} \theta(0) dx \geq 0$$

$$\sup_{t \in (0, T)} \int_{\mathbb{T}^3} \rho dx \leq K_1, \quad \sup_{t \in (0, T)} \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|m|^2}{\rho} + h(\rho) \right] dx \leq K_2$$

the pressure $p(\rho)$ satisfies $p''(\rho) \leq A \frac{p'(\rho)}{\rho} \forall \rho > 0$; for instance, $p(\rho) = \rho^\gamma, \gamma \geq 1$

Stability and convergence/1

We denote by

$$\varphi(t) = \int_{\mathbb{T}^3} \eta(\rho, m | \bar{\rho}, \bar{m}) dx$$

Theorem

Let $T > 0$ be fixed. Under hypothesis (\mathbf{H}_1) , the following stability estimate holds:

$$\varphi(t) \leq C(\varphi(0) + \varepsilon^4), \quad t \in [0, T],$$

where C is a positive constant depending only on T , K_1 , $\bar{\rho}$ and its derivatives.

Moreover, if $\varphi(0) \rightarrow 0$ as $\varepsilon \downarrow 0$, then

$$\sup_{t \in [0, T]} \varphi(t) \rightarrow 0, \quad \text{as } \varepsilon \downarrow 0$$

Stability and convergence/2

Proof.

$$\varphi(t) + \frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^3} R(\rho, m | \bar{\rho}, \bar{m}) d\tau dx \leq \varphi(0) + \int_0^t \int_{\mathbb{T}^3} (|Q| + |E|) d\tau dx$$

$$\int_0^t \int_{\mathbb{T}^3} |Q| d\tau dx \leq C_0 \int_0^t \varphi(\tau) d\tau$$

$$\begin{aligned} \int_0^t \int_{\mathbb{T}^3} |E| d\tau dx &\leq \frac{\varepsilon^2}{2} \int_0^t \int_{\mathbb{T}^3} \left| \frac{e(\bar{\rho}, \bar{m})}{\bar{\rho}} \right|^2 \rho d\tau dx + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^3} \rho \left| \frac{m}{\rho} - \frac{\bar{m}}{\bar{\rho}} \right|^2 d\tau dx \\ &\leq \varepsilon^4 t C_1 K_1 + \frac{1}{2\varepsilon^2} \int_0^t \int_{\mathbb{T}^3} R(\rho, m | \bar{\rho}, \bar{m}) d\tau dx \end{aligned}$$

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$$\left\{ \begin{array}{l} \rho_t + \frac{1}{\varepsilon} \operatorname{div}_x m = 0 \\ m_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{m \otimes m}{\rho} + \frac{1}{\varepsilon} \nabla_x p(\rho) = -\frac{1}{\varepsilon^2} m + \frac{1}{\varepsilon} \rho \nabla_x c \\ -\Delta_x c + \beta c = \rho, \end{array} \right.$$

where $\rho \geq 0$, $c \in \mathbb{R}$, $m \in \mathbb{R}^3$, the pressure $p(\rho)$ satisfies $p'(\rho) \geq 0$ and $\beta > 0$. Easiest case $p(\rho) = \kappa \rho^2$, $\beta > 1/2\kappa$.

Formal limit:

$$\left\{ \begin{array}{l} \rho_t + \operatorname{div}_x (\rho \nabla_x c - \nabla_x p(\rho)) = 0 \\ -\Delta_x c + \beta c = \rho \end{array} \right.$$

Entropy (in)equalities

Modified entropy–entropy flux pair, based on the mechanical energy of the system:

$$\begin{aligned}\mathcal{H}(\rho, m, c) &= \eta(\rho, m) - \rho c \\ \mathcal{Q}(\rho, m, c) &= q(\rho, m) - mc.\end{aligned}$$

Then the entropy inequality becomes

$$\mathcal{H}(\rho, m, c)_t + \frac{1}{\varepsilon} \operatorname{div}_x \mathcal{Q}(\rho, m, c) \leq -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho} - \rho c_t$$

From the elliptic equation:

$$\rho c_t = \frac{1}{2} (\beta c^2 + |\nabla_x c|^2)_t - \operatorname{div}_x (c_t \nabla_x c)$$

Final relation:

$$\left(\mathcal{H}(\rho, m, c) + \frac{1}{2} (\beta c^2 + |\nabla_x c|^2) \right)_t + \frac{1}{\varepsilon} \operatorname{div}_x (\mathcal{Q}(\rho, m, c) - \varepsilon c_t \nabla_x c) \leq -\frac{1}{\varepsilon^2} \frac{|m|^2}{\rho}$$

Relative entropy estimate

$(\bar{\rho}, \bar{m}, \bar{c}) = (\bar{\rho}, -\varepsilon \bar{\rho} \nabla_x (h'(\bar{\rho}) - \bar{c}), \bar{c})$ solves

$$\begin{cases} \bar{\rho}_t + \frac{1}{\varepsilon} \operatorname{div}_x \bar{m} = 0 \\ \bar{m}_t + \frac{1}{\varepsilon} \operatorname{div}_x \frac{\bar{m} \otimes \bar{m}}{\bar{\rho}} + \frac{1}{\varepsilon} \nabla_x p(\bar{\rho}) = -\frac{1}{\varepsilon^2} \bar{m} + \frac{1}{\varepsilon} \bar{\rho} \nabla_x \bar{c} + e(\bar{\rho}, \bar{m}) \\ -\Delta_x \bar{c} + \beta \bar{c} = \bar{\rho} \end{cases}$$

Relative entropy estimate:

$$\begin{aligned} & \left(\mathcal{H}(\rho, m, c | \bar{\rho}, \bar{m}, \bar{c}) + \frac{1}{2} (\beta(c - \bar{c})^2 + |\nabla_x(c - \bar{c})|^2) \right)_t \\ & + \frac{1}{\varepsilon} \operatorname{div}_x \left(\mathcal{Q}(\rho, m, c | \bar{\rho}, \bar{m}, \bar{c}) - \varepsilon(c - \bar{c})_t \nabla_x(c - \bar{c}) \right) \\ & \leq -\frac{1}{\varepsilon^2} R(\rho, m | \bar{\rho}, \bar{m}) - Q - P - E, \end{aligned}$$

where R , Q and E are as before and $P = \frac{1}{\varepsilon} \frac{\bar{m}}{\bar{\rho}} (\rho - \bar{\rho}) \cdot \nabla_x(c - \bar{c})$

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The model

$$\begin{cases} u_t - \frac{1}{\varepsilon} v_x = 0 \\ v_t - \frac{1}{\varepsilon} \tau(u)_x = -\frac{1}{\varepsilon^2} v, \end{cases}$$

where τ satisfies the usual conditions $\tau'(u) > 0$ to guarantee strict hyperbolicity. For gas dynamics applications, u denotes the specific volume, v the velocity and

$$\tau\left(\frac{1}{\rho}\right) = -p(\rho),$$

where p stands for the pressure of the gas and ρ for its density.
Formal limit

$$u_t - \tau(u)_{xx} = 0,$$

thus with the relation (Darcy's law) at the limit $v = \tau(u)_x$.

Entropy (in)equalities

$$\mathcal{E}(u, v) = \frac{1}{2}v^2 + \int^u \tau(s)ds = \frac{1}{2}v^2 + W(u),$$

with entropy flux given by

$$\mathcal{F}(u, v) = -v\tau(u)$$

and corresponding entropy inequality

$$\mathcal{E}(u, v)_t + \frac{1}{\varepsilon} \mathcal{F}(u, v)_x \leq -\frac{1}{\varepsilon^2} v^2 \leq 0$$

$\mathcal{E}(u, 0) = W(u)$ entropy for the limiting equation:

$$\mathcal{E}(u, 0)_t + \mathcal{F}(u, \tau(u)_x)_x = -(\tau(u)_x)^2$$

Relative entropy estimate

$(\bar{u}, \bar{v}) = (\bar{u}, \varepsilon \tau(\bar{u})_x)$ solves

$$\begin{cases} \bar{u}_t - \frac{1}{\varepsilon} \bar{v}_x = 0 \\ \bar{v}_t - \frac{1}{\varepsilon} \tau(\bar{u})_x = -\frac{1}{\varepsilon^2} \bar{v} + \varepsilon \tau(\bar{u})_{xt} \end{cases}$$

Relative entropy:

$$\mathcal{E}(u, v | \bar{u}, \bar{v}) = \frac{1}{2}(v - \bar{v})^2 + W(u | \bar{u})$$

$$\begin{aligned} \mathcal{E}(u, v | \bar{u}, \bar{v})_t + \frac{1}{\varepsilon} \mathcal{F}(u, v | \bar{u}, \bar{v})_x &\leq -\frac{1}{\varepsilon^2}(v - \bar{v})^2 + \tau(\bar{u})_{xx} \tau(u | \bar{u}) \\ &\quad - \varepsilon \tau(\bar{u})_{xt} (v - \bar{v}) \end{aligned}$$

Outline

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The model

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x - \frac{1}{\varepsilon} z_x = 0 \\ z_t - \frac{\mu}{\varepsilon} v_x = -\frac{1}{\varepsilon^2} z, \end{cases}$$

where $\mu > 0$ and the elastic stress function σ satisfies the usual condition $\sigma'(u) > 0$ which guarantees strict hyperbolicity.

Formal limit:

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \mu v_{xx}, \end{cases}$$

for which the viscoelastic response is given by $z = \sigma(u) + \mu v_x$

Entropy (in)equalities

$$\mathbb{E}(u, v, z) = \int^u \sigma(s) ds + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2 = \Sigma(u) + \frac{1}{2}v^2 + \frac{1}{2\mu}z^2,$$

with entropy flux given by

$$\mathbb{F}_\varepsilon(u, v, z) = -(\varepsilon\sigma(u)v + vz)$$

and corresponding entropy inequality

$$\mathbb{E}(u, v, z)_t + \frac{1}{\varepsilon}\mathbb{F}_\varepsilon(u, v, z)_x \leq -\frac{1}{\mu\varepsilon^2}z^2 \leq 0$$

$\mathbb{E}(u, v, 0) = \Sigma(u) + \frac{1}{2}v^2$ entropy for the limiting system:

$$\mathbb{E}(u, v, 0)_t + \mathbb{F}_1(u, v, \sigma(u)_x)_x = -\mu(v_x)^2$$

for

$$\mathbb{F}_1(u, v, \sigma(u)_x) = -(\sigma(u)v + \mu v v_x)$$

Relative entropy estimate

$(\bar{u}, \bar{v}, \bar{z}) = (\bar{u}, \bar{v}, \varepsilon\mu\bar{v}_x)$ solves

$$\begin{cases} \bar{u}_t - \bar{v}_x = 0 \\ \bar{v}_t - \sigma(\bar{u})_x - \frac{1}{\varepsilon}\bar{z}_x = 0 \\ \bar{z}_t - \frac{\mu}{\varepsilon}\bar{v}_x = -\frac{1}{\varepsilon^2}\bar{z} + \varepsilon\mu\bar{v}_{xt} \end{cases}$$

Relative entropy:

$$\mathbb{E}(u, v, z | \bar{u}, \bar{v}, \bar{z}) = \Sigma(u | \bar{u}) + \frac{1}{2}(v - \bar{v})^2 + \frac{1}{2\mu}(z - \bar{z})^2$$

$$\begin{aligned} \mathbb{E}(u, v, z | \bar{u}, \bar{v}, \bar{z})_t + \frac{1}{\varepsilon}\mathbb{F}_\varepsilon(u, v, z | \bar{u}, \bar{v}, \bar{z})_x &\leq -\frac{1}{\mu\varepsilon^2}(z - \bar{z})^2 + \bar{v}_x\sigma(u | \bar{u}) \\ &\quad - \varepsilon\bar{v}_{xt}(z - \bar{z}) \end{aligned}$$

THANK YOU