

Tunneling for continuous time Markov processes.
(Joint work with Claudio Landim)

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General setting:

For each $N \geq 1$, let $(\eta_t^N)_{t \geq 0}$ be an irreducible Markov process on a countable state space E_N .

$$\mathbf{W} = (W_N \subseteq E_N : N \geq 1), \quad \mathbf{B} = (B_N \subseteq E_N : N \geq 1)$$

$$W_N \subseteq B_N \subsetneq E_N.$$

Fix $\boldsymbol{\xi} = (\xi_N \in W_N : N \geq 1)$ and $\boldsymbol{\theta} = (\theta_N : N \geq 1)$. Denote $\Delta_N = B_N \setminus W_N$.

Definition

$(\mathbf{W}, \mathbf{B}, \boldsymbol{\xi})$ is a valley of depth $\boldsymbol{\theta}$ and attractor $\boldsymbol{\xi}$ if

1. $T_{B^c}/\theta_N \rightarrow \exp(1)$ in law, under \mathbb{P}_{η^N} for any $\eta^N \in W_N$.
2. For every $\delta > 0$,

$$\sup_{\eta \in W_N} \mathbb{P}_{\eta} \left[\frac{1}{\theta_N} \int_0^{T_{B^c}} \mathbf{1}_{\{\eta_s^N \in \Delta_N\}} ds > \delta \right] \rightarrow 0,$$

3. $\inf_{\eta \in W_N} \mathbb{P}_{\eta} [T_{\boldsymbol{\xi}} < T_{B^c}] \rightarrow 1$;

Let us assume that (η_t^N) is positive recurrent and its invariant measure ν_N is reversible.

Given $\xi \in \mathbf{W}$, define

$$\text{cap}_N(\xi) := \inf\{\text{cap}_N(\eta, \xi) : \eta \in W_N\}$$

Theorem

If

- For some $\xi \in \mathbf{W}$,

$$\lim_{N \rightarrow \infty} \frac{\text{cap}_N(\mathbf{W}, \mathbf{B}^c)}{\text{cap}_N(\xi)} = 0$$

- $\nu_N(\Delta_N)/\nu_N(W_N) \rightarrow 0$

then $(\mathbf{W}, \mathbf{B}, \xi)$ is a valley with attractor $\xi \in \mathbf{W}$ and depth

$$\theta_N = \frac{\nu_N(\mathbf{W})}{\text{cap}_N(\mathbf{W}, \mathbf{B}^c)}.$$

Example

Zero-range process:

- ▶ Fix a finite set of sites $S = \{1, 2, \dots, \kappa\}$ with $\kappa \geq 2$.
- ▶ For each $N \geq 1$, we consider the set of configurations

$$E_N = \left\{ \eta \in \mathbb{Z}_+^S : \sum \eta_x = N \right\}$$

- ▶ Given $g : \mathbb{Z}_+ \mapsto \mathbb{R}_+$ with $g(0) = 0$, and $\overset{x \in S}{p} = \{p(x, y) : x, y \in S\}$

A (g, p) -zero-range is a Markov process with generator

$$L_N f(\eta) = \sum_{x, y \in S} g(\eta_x) p(x, y) [f(\sigma^{x, y} \eta) - f(\eta)]$$

Our zero-range:

- ▶ Fix $\alpha > 1$. Let $g(1) = 1$ and

$$g(n) = \left(1 + \frac{1}{n-1} \right)^\alpha, \quad \forall n \geq 2$$

$(g(n) : n \geq 2)$ is a strictly decreasing sequence converging to 1.

Complete graph

Suppose that $p(x, y) = 1/(\kappa - 1)$ for every $x \neq y$.

- ▶ The invariant measure:

$$\nu_N(\eta) = \left(\frac{N^\alpha}{Z_N} \right) \frac{1}{(\eta_1)^\alpha (\eta_2)^\alpha \dots (\eta_\kappa)^\alpha}, \quad \forall \eta \in E_N$$

where $Z_N > 0$ is a normalizing constant of order 1.

- ▶ For each $x \in S$ we define the subsets

$$\mathcal{E}_N^x = \{ \eta \in E_N : \eta_x \geq N - \ell_N \}$$

where $\ell_N \rightarrow \infty$ and $\ell_N/N \rightarrow 0$.

The sequence ℓ_N determines the partition

$$E_N = \underbrace{\mathcal{E}_N^1 \cup \mathcal{E}_N^2 \cup \dots \cup \mathcal{E}_N^\kappa}_{\mathcal{E}_N} \cup \Delta_N$$

$$\nu_N(\mathcal{E}_N^x) \rightarrow 1/\kappa \quad \text{and} \quad \nu_N(\Delta_N) \rightarrow 0.$$

Let $\xi \in \mathcal{E}^x$ be any sequence $\xi^N \in \mathcal{E}_N^x$.

Theorem

If $\ell_N \rightarrow \infty$ and

$$\lim_{N \rightarrow \infty} \frac{(\ell_N)^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} = 0,$$

then each $(\mathcal{E}^x, \mathcal{E}^x \cup \Delta, \xi)$ is a valley of depth $c_{\kappa, \alpha} N^{\alpha+1}$, where $c_{\kappa, \alpha}$ is a constant depending only on α and κ .

General setting:

- ▶ For each $N \geq 1$, let $(\eta_t^N)_{t \geq 0}$ be an irreducible Markov process on a countable state space E_N .
- ▶ Fix $\kappa \geq 2$. Denote $S = \{1, 2, \dots, \kappa\}$. For all $N \geq 1$,

$$E_N = \underbrace{\mathcal{E}_N^1 \cup \mathcal{E}_N^2 \cup \dots \cup \mathcal{E}_N^\kappa}_{\mathcal{E}_N} \cup \Delta_N$$

Let $\Psi_N : \mathcal{E}_N \mapsto S$ be given by

$$\Psi_N(\eta) = \sum_{x \in S} x \mathbf{1}\{\eta \in \mathcal{E}_N^x\}.$$

- ▶ $X_t^N := \Psi_N(\tilde{\eta}_t^N)$, where $\tilde{\eta}_t^N$ is the trace of η_t^N on \mathcal{E}_N .

Trace process:

Define

$$\mathcal{T}_t^N := \int_0^t \mathbf{1}_{\{\eta_s^N \in \mathcal{E}_N\}} ds$$

and

$$\mathcal{S}_t^N(\omega) := \sup\{s \geq 0 : \mathcal{T}_s^N \leq t\}.$$

The trace of the Markov process η_t^N on \mathcal{E}_N is

$$\tilde{\eta}_t^N := \eta_{\mathcal{S}_t^N}^N.$$

In this way, we obtain a Markov process $(\tilde{\eta}_t^N)$ on \mathcal{E}_N .

Definition (Tunneling)

We say that $(\eta_t^N)_{t \geq 0}$ exhibits tunneling on the time-scale θ_N if

(A) For every time $t \geq 0$

$$\lim_{N \rightarrow \infty} \sup_{\eta \in \mathcal{E}_N} \mathbb{E}_\eta^N \left[\int_0^t \mathbf{1}\{ \eta_{s\theta_N}^N \in \Delta_N \} ds \right] = 0 .$$

(B) $X_{t\theta_N}^N$ converges in law to a Markov process on $S = \{1, \dots, \kappa\}$

Suppose that $\ell_N \rightarrow \infty$ and

$$\lim_{N \rightarrow \infty} \frac{(\ell_N)^{1+\alpha(\kappa-1)}}{N^{1+\alpha}} = 0.$$

Theorem (Complete graph)

If $p(x, y) = 1/(\kappa - 1)$ then our condensed zero-range process exhibits tunneling on time-scale $N^{1+\alpha}$ and X_t^N converges to a random walk on the complete graph with set of vertices S .

Theorem (Nearest neighbor)

If $p(x, x + 1) = 1/2$ for $x \leq \kappa - 1$, $p(x, x - 1) = 1/2$ for $x \geq 1$ and $p(0, 0) = p(\kappa, \kappa) = 1/2$, then our condensed zero-range process exhibits tunneling on time-scale $N^{1+\alpha}$ and X_t^N converges to a Markov chain on S with rates:

$$r(x, y) = \lambda_{\kappa, \alpha} (|x - y|)^{-1},$$

for some constant $\lambda_{\kappa, \alpha}$ depending only on κ and α .

Reversible $p(x, y)$

Assume that $\{p(x, y) : x, y \in S\}$ is irreducible and the invariant measure $m : S \rightarrow [0, 1]$ is reversible (i.e. $m(x)p(x, y) = m(y)p(y, x)$). Denote

$$m_* := \max\{m(x) : x \in S\} .$$

Let $S_* = \{x \in S : m(x) = m_*\}$ and $\kappa = |S_*|$. Suppose that $\kappa \geq 2$.

Theorem

There exists some $\epsilon_0 > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, by letting $\ell_N = N^\epsilon$, our condensed zero-range process exhibits tunneling on time-scale $N^{1+\alpha}$ and X_t^N converges to a Markov chain on S with rates:

$$r(x, y) = \beta_{\kappa, \alpha} \text{cap}_S(x, y) ,$$

where $\text{cap}_S(x, y)$ are the capacities associated to m and $\{p(x, y) : x, y \in S\}$.