Amplitude Equations

natural slow-fast systems for SPDEs

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Key Idea:

Complicated models near a change of stability

$\Rightarrow$ slow-fast system

- Dominant pattern evolve on a *slow time-scale*
- Stable pattern decay/disappear on a *fast time-scale*

Dynamics described by simplified model

$\Rightarrow$ Amplitude Equations – Averaging

Deterministic Results for PDEs

- PDEs[1] on bounded domains:
  Invariant manifolds – center manifold
  Solutions approximated by an ODE on the manifold
- PDEs on unbounded domains:
  Many rigorous results since the '90-th
  (e.g. Schneider, Uecker, Collet, Eckmann, ... & many others)
  Validation of Amplitude or Modulation equations

Our Problem:

- What is the impact of noise on the dominant pattern?
- No useful center manifold theory for stochastic PDEs yet

[1] PDE=partial differential equations
SPDE=stochastic PDE

Our Aim

AIM:

- Validation of amplitude equations for SPDEs
- Understand effect of noise transported by the nonlinearity

For the talk only one example:

- Swift-Hohenberg model
- noise constant in space or space-time white

Swift-Hohenberg is a celebrated model in pattern formation.

The first change of stability is a toy model for the convective instability in Rayleigh-Bénard convection.

Slow-Fast System I

SPDE close to bifurcation naturally generate slow-fast systems

SDE coupled to a fast SPDE

Example:

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SPDE)}$$

- $\mathcal{L}$ non-positive operator - kernel $\mathcal{N}$ (dominant modes)
- $\nu\epsilon^2$ the distance from bifurcation
- ϵ^2 noise strength
- $\{W(t)\}_{t\geq 0}$ some Wiener process (∞-dimensional)

Slow-Fast System II

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SPDE)}$$

Split

$$u(t) = \epsilon v_c(\epsilon^2 t) + \epsilon v_s(\epsilon^2 t)$$

with $v_c \in \mathcal{N}$ and $v_s \perp \mathcal{N}$ ($\mathcal{N}$ kernel of $\mathcal{L}$)

$$\partial_T v_c = \nu v_c - P_c(v_c + v_s)^3 + \partial_T \tilde{W}_c \qquad \text{(SLOW)}$$

$$\partial_T v_s = \epsilon^{-2}\mathcal{L}v_s + \nu v_s - P_s(v_c + v_s)^3 + \partial_T \tilde{W}_s \qquad \text{(FAST)}$$

where:

P_c projects onto $\mathcal{N}$ and $P_s = I - P_c$

$\tilde{W}(T) = \epsilon W(T\epsilon^{-2})$ rescaled Wiener process

Outline

Consider Swift-Hohenberg equation only.

1. Brief review – full noise – bounded domains

   [DB, Maier-Paape, Schneider 01], [DB, Hairer 04,05]

2. Relation to random invariant manifolds
3. Stabilization by degenerate noise

   [Hutt, et.al., 07,08], [DB, Mohammed 10], [Roberts 03], [DB, Hairer, Pavliotis 07]

4. Work in progress on unbounded domains

   [DB, Klepel, Mohammed]

5. Outlook / Open Problems

Swift-Hohenberg Equation

Swift-Hohenberg

[DB, Maier-Paape, Schneider 01], [DB, Hairer, 04]

Swift-Hohenberg equation

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon^2 \partial_t W \qquad \text{(SH)}$$

with

- $\mathcal{L} = -(1+\partial_x^2)^2$
- periodic boundary conditions on $[0, 2\pi]$
- dominant modes $\mathcal{N} = \text{span}\{\sin, \cos\}$
- space-time white noise $\partial_t W$

Existence & Uniqueness

Local solutions via fixed-point arguments:

mild solutions – variation of constants

$$u(t) = e^{t\mathcal{L}}u(0) + \int_0^t e^{(t-s)\mathcal{L}} \left[\nu\epsilon^2 u(s) - u^3(s)\right] ds + W_A(t)$$

where the stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)\mathcal{L}} dW(s)$$

and its spatial derivative $\partial_x W_A$ is in $C^0([0,T] \times [0,2\pi])$.

Theorems
Attractivity & Approximation

Theorem (Attractivity) [DB, Hairer 04]

For all $T > 0$: $u_c(T\epsilon^{-2}) = \mathcal{O}(\epsilon)$ and $u_s(T\epsilon^{-2}) = \mathcal{O}(\epsilon^2)$

Theorem (Approximation) [DB, Hairer 04]

$u(0) = \epsilon a(0) + \epsilon^2 \psi(0)$ with $a(0)$ and $\psi(0)$ both $\mathcal{O}(1)$.
Let $a(T) \in \mathcal{N}$ solve

$$\partial_T a = \nu a - P_c a^3 + \partial_T \tilde{W}_c$$

and $\psi(t) \perp \mathcal{N}$ solves $\partial_t \psi = \mathcal{L}\psi + \partial_t W_s$

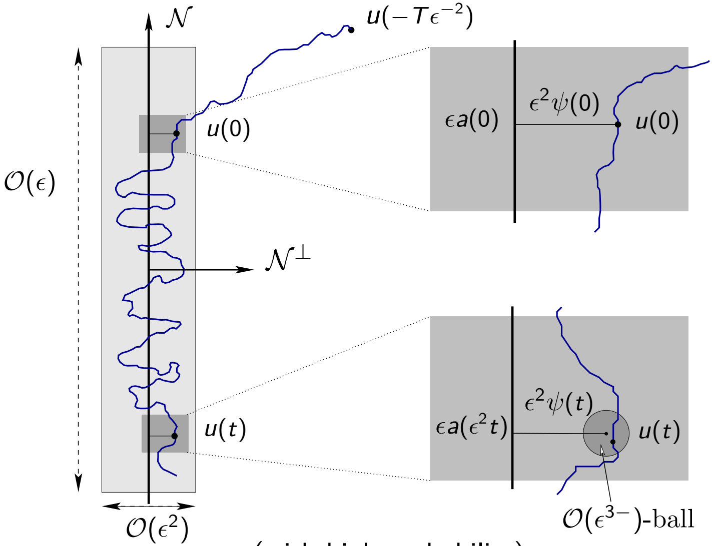
Then for $t \in [0, T_0\epsilon^{-2}]$

$$u(t) = \epsilon a(\epsilon^2 t) + \epsilon^2 \psi(t) + \mathcal{O}(\epsilon^{3-})$$

Remark: Approximation remains true for invariant measures.

Approximative Center Manifold

[DB, Hairer 05]



(with high probability)

Random Dynamical Systems

Random Invariant Manifolds

Random Dynamical System (RDS)

[L. Arnold, Crauel, Schmalfuß, Flandoli, Scheutzow, Chueshov, Duan, Caraballo, Kloeden, Robinson,....]

Solutions generate a random dynamical system

$\varphi(t, W)u_0$ is solution of (SPDE) given

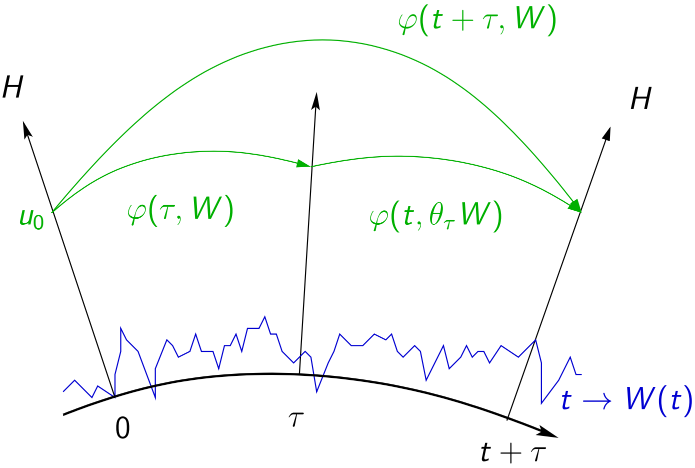
- $t \geq 0$ – time
- u_0 – initial condition
- W – whole path of the Wiener process $t \to W(t)$, $t \in \mathbb{R}$.

Cocycle property: For all $t, \tau \in \mathbb{R}^+$ and paths W.

$$\varphi(0, W) = Id, \qquad \varphi(t + \tau, W) = \varphi(t, \theta_\tau W)\varphi(\tau, W)$$

Ergodic Shift: $\theta_\tau W = W(t + \cdot) - W(t)$

Cocycle Property $\varphi(t, \theta_\tau W)\varphi(\tau, W) = \varphi(t+\tau, W)$



Random Invariant Manifold (RIM)

[Duan, Lu, Schmalfuß 03, 04], [Mohammed, Zhang, Zhao, 08]

Definition (Random Invariant Manifold, RIM)

A random set $M(W)$ is positive invariant, if

$$\varphi(t, W)M(W) \subset M(\theta_t W) \text{ for all } t \geq 0.$$

If

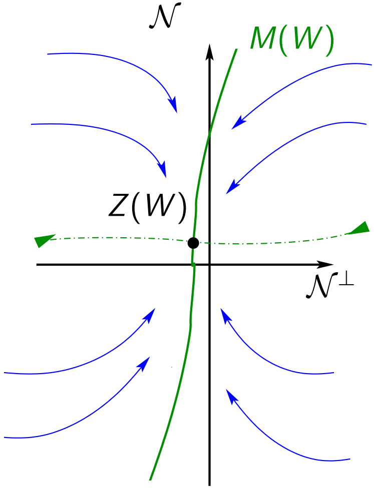
$$M(W) = \{u + \psi(W, u) | u \in \mathcal{N}\}$$

is the graph of a random Lipschitz mapping

$$\psi(W, \cdot) : \mathcal{N} \to \mathcal{N}^\perp ,$$

then $M(W)$ is called a Lipschitz RIM.

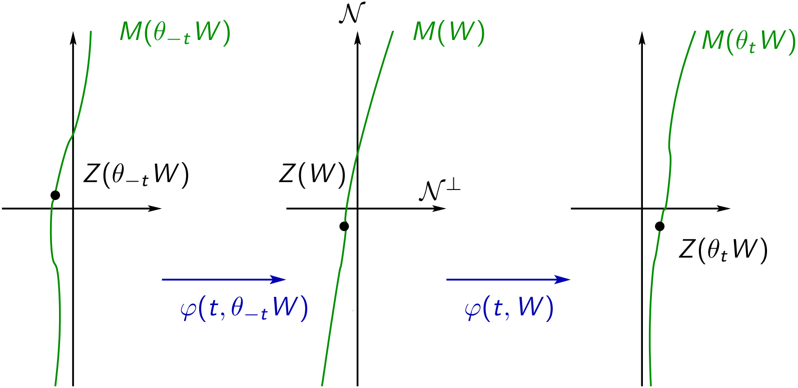
stable RIM for Swift Hohenberg



for $\nu < 0$

- $Z(W)$ random fixed point
- $t \mapsto Z(\theta_t W)$
  stable stationary solution
- $M(W)$
  random invariant manifold
- convergence in pull-back sense (initial time to $-\infty$)

RIM are moving in time!



Summary: RIM vs Amplitude Eq.

Random invariant manifolds:

- behaviour for all paths
- reduced SDE on a moving space

Amplitude Equations:

- only the most likely behaviour
- reduced SDE on a fixed vector space

Useful for the derivation of qualitative properties of RIM & invariant measures.

Stabilization due to Noise

Interesting observation:

Additive**(!)** noise may lead to stabilization (or shift of bifurcation) of dominant modes (pattern disappears)

Stabilization due to Noise

Well known phenomenon due to multiplicative Noise. For example:

By Itô noise, due to Itô-Stratonovic correction, or Stratonovic noise due to averaging over stable and unstable directions

- **For SDE:** [Arnold, Crauel, Wihstutz '83], [Pardoux, Wihstutz '88 '92].....
- **For SPDE:** [Kwiecinska '99], [Caraballo, Mao et.al. '01], [Cerrai '05], [Caraballo, Kloeden, Schmallfuß '06]....

By Rotation: [Baxendale et.al.'93], [Crauel et.al.'07].....

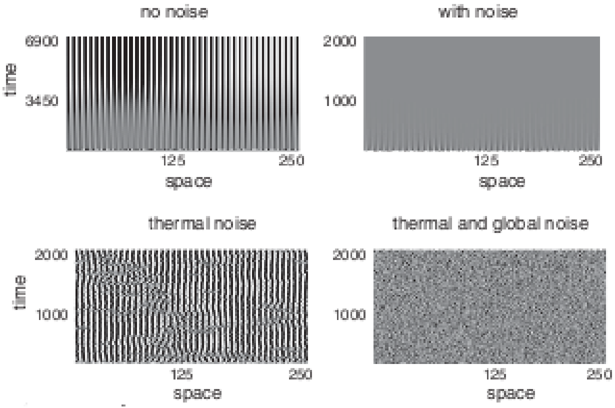
Only very few examples due to additive noise:

Blow up through a small tube: e.g. [Scheutzow et.al.'93]...

Numerical Examples

Swift-Hohenberg-model

Space-independent (gobal) noise destroys modulated pattern in 1D-Swift-Hohenberg-Eq.



Swift-Hohenberg Equation

Example: Swift-Hohenberg Equation

$$\partial_t u = -(1+\partial_x^2)^2 u + \tfrac{1}{20} u - u^3 + \tfrac{\sigma}{10}\partial_t \beta(t) \qquad \text{(SH)}$$

- $u(t,x) \in \mathbb{R}$, $t > 0$, $x \in [0, 2\pi]$
- periodic boundary conditions

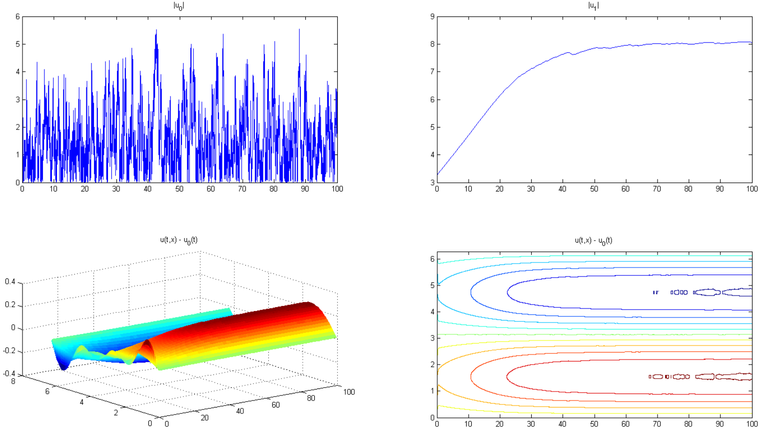
Observation:

- 0 is is stabilized by large noise
- Noise only in time, constant in space

Swift-Hohenberg

(Nolde, Wöhrl) [DB, Nolde, Mohammed, Wöhrl, 12]

$\sigma = 0.5$

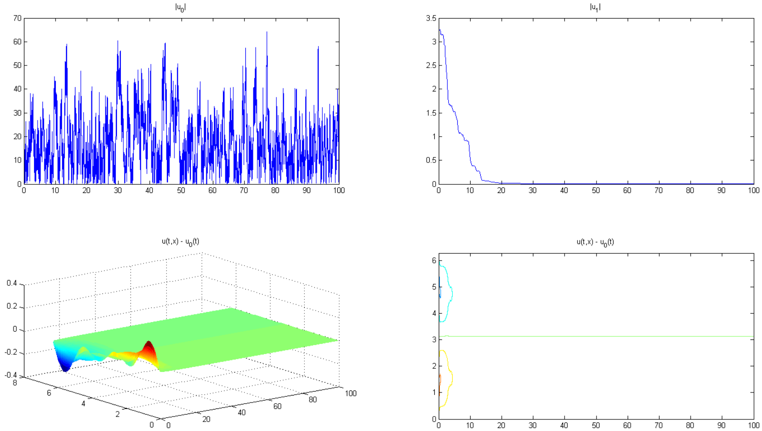


Semi-implicit spectral Galerkin-method using fft in Matlab

Swift-Hohenberg

(Nolde, Wöhrl) [DB, Nolde, Mohammed, Wöhrl, 12]

$\sigma = 5$



Semi-implicit spectral Galerkin-method using fft in Matlab

Results for Swift-Hohenberg Equation

The SPDE – Swift-Hohenberg

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\,\partial_t\beta \qquad \text{(SH)}$$

- $\mathcal{L} = -(1+\partial_x^2)^2$
- periodic boundary conditions on $[0, 2\pi]$
- $\text{span}\{\sin, \cos\}$ – dominant pattern
- β is a real-valued Brownian motion
- $\sigma\epsilon \ll 1$ – noise strength
- $|\nu|\epsilon^2 \ll 1$ – distance from bifurcation

The Ansatz

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \epsilon\sigma\partial_t\beta \qquad \text{(SH)}$$

Ansatz:

$$u(t,x) = \epsilon A(\epsilon^2 t)\mathrm{e}^{ix} + c.c. + \epsilon\sigma \cdot Z(t) + \mathcal{O}(\epsilon^2)$$

fast process – $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$

complex-valued amplitude – A

Result: Amplitude Equation [DB, Mohammed, '10]

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

Interesting Facts

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

- Amplitude equation is deterministic
- Noise leads to a stabilizing deterministic correction

Formal calculation with SPDEs yields:

New contribution to the amplitude equation is

$$3A \cdot \sigma^2 |\epsilon \partial_T \tilde{\beta}|^2 \ ,$$

where $\tilde{\beta}(T) = \epsilon\beta(T\epsilon^{-2})$
is a Brownian motion on the slow time-scale $T = \epsilon^2 t$

Noise^2?

What is noise^2?

Instead of $\epsilon\partial_T\tilde\beta$ consider a fast Ornstein-Uhlenbeck-process

$$Z(T\epsilon^{-2}) = Z_\epsilon(T) = \epsilon^{-1}\int_0^T e^{-(T-s)\epsilon^{-2}} d\tilde\beta(s) \approx \epsilon\partial_T\tilde\beta(T)\ ,$$

where $\tilde\beta(T) = \epsilon\beta(\epsilon^{-2}T)$ is a rescaled Brownian motion

[DB, Hairer, Pavliotis, '07] [DB, Mohammed '08] Averaging with error bounds

$dX = \mathcal{O}(\epsilon^{-r})dt + \mathcal{O}(\epsilon^{-r})d\tilde\beta$ and $X(0) = \mathcal{O}(\epsilon^{-r})$, $r > 0$, then

$$\int_0^T X(s)Z_\epsilon(s)^2 ds = \tfrac{1}{2}\int_0^T X(s)ds + \mathcal{O}(\epsilon^{1-2r})$$

Averaging

other averaging results

For odd powers:

$$\int_0^T X(s) Z_\epsilon(s) ds = \mathcal{O}(\epsilon^{1-r})$$

$$\int_0^T X(s) Z_\epsilon(s)^3 ds = \mathcal{O}(\epsilon^{1-3r})$$

and so on ...

Note: $X = \mathcal{O}(f_\epsilon)$
if $\forall\ p > 1,\ T > 0$ there is a $C > 0$ such that

$$\mathbb{E} \sup_{[0,T]} |X|^p \leq C f_\epsilon^p$$

The Theorem

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{(SH)}$$

$$\partial_T A = (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(A)}$$

Theorem – Approximation [DB, Mohammed '10]

u is solution of (SH) in H^1 – A is solution of (A)
$u(0) = \epsilon\, A(0)\mathrm{e}^{ix} + c.c. + \epsilon\psi_0$ with $\psi_0 \perp \mathrm{e}^{\pm ix}$.

Then for κ, T_0, $p > 0$ there is $C > 0$ such that

$$\mathbb{P}\Big(\sup_{t\in[0,T_0/\epsilon^2]} \|u(t) - \epsilon v(\epsilon^2 t)\|_{H^1} > \epsilon^{2-\kappa} \Big)$$
$$< C\epsilon^p + \mathbb{P}(\|u(0)\|_{H^1} > \epsilon^{1-\kappa/2})$$

with $v(T) = A(T)\mathrm{e}^{ix} + c.c. + \sigma Z_\epsilon(T) + e^{T\epsilon^{-2}\mathcal{L}}\psi_0$.

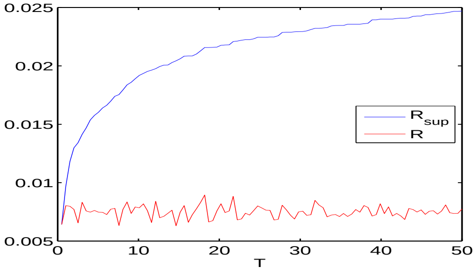
Remark: Theorem holds in a much more general setting.

Numerical justification

(Nolde / Wöhrl) [DB, Nolde, Mohammed, Wöhrl, 12]

Numerical approximation of $R_{\text{sup}}(T) = \sup_{s\in[0,T]} R(s)$ and

$$R(T) = \left\| u(T\epsilon^{-2}) - \epsilon \left[Z_\epsilon(T) + A(T)e^{ix} + c.c. \right] \right\|_\infty$$



$\nu = 1$, $\sigma = \frac{1}{\sqrt{15}}$, $\epsilon = \frac{1}{10}$ – mean over a few hundred realizations

Open Problem

If we consider higher order corrections [DB, Mohammed, '10]

or Burgers equation & degenerate noise [DB, Hairer, Pavliotis, 07]

we need to treat terms like

$$\int_0^T X(s) Z_\epsilon(s)^2 d\tilde{\beta}(s)$$

Problem:

In order to derive *error estimates* we rely on Martingal representation and Levy characterization.

Thus $\dim(\mathcal{N}) = 1$ is necessary ↯

For $\dim(\mathcal{N}) > 1$ weak convergence results available

$\Longrightarrow$ **Averaging** (well known)

Martingale Approximation Result in 1D

For $\int_0^T f(a, Z_\epsilon) d\tilde{\beta}$ with $Z_\epsilon(t) = \frac{1}{\epsilon} \int_0^t e^{-(t-s)\lambda\epsilon^{-2}} d\tilde{\beta}(s)$.

Lemma [DB, Hairer, Pavliotis, '07]

$M(t)$ continuous martingale with quadratic variation f

g arbitrary adapted increasing process with $g(0) = 0$

Then (with respect to an enlarged filtration) there exists a continuous martingale $\tilde{M}(t)$ with quadratic variation g

such that $\forall\, \gamma < 1/2 \;\; \exists\, C > 0$ with

$$\mathbb{E} \sup_{[0,T]} |M - \tilde{M}|^p \;\leq\; C\, \mathbb{E} \sup_{[0,T]} |f - g|^{p/2} + C \big(\mathbb{E} g(T)^{2p}\big)^{1/4} \big(\mathbb{E} \sup_{[0,T]} |f - g|^p\big)^\gamma .$$

Unbounded or Large Domains

Modulated Pattern

Problem:

A full band of eigenvalues changes stability

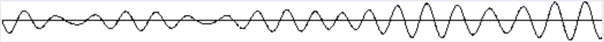
Modulation of dominant pattern

Periodic Pattern

$u(x) \approx \epsilon A e^{ix} + c.c.$ where $A \in \mathbb{C}$

Modulated Pattern (many modes near ± 1 contribute)

$u(x) \approx \epsilon A(\epsilon x) e^{ix} + c.c.$ where $A : \mathbb{R} \mapsto \mathbb{C}$



Results on Large Domains

$$\partial_t u = \mathcal{L}u + \epsilon^2 \nu u - u^3 + \epsilon^{\frac{3}{2}} \xi \qquad (SH)$$

Results for deterministic PDE (on $\mathbb{R}$):

[Kirrmann, Mielke, Schneider, 92] and many more ...

Result for SPDE (on $[-L/\epsilon, L/\epsilon]$):

Only on large domains [DB, Hairer, Pavliotis '05]

$$u(t,x) = \epsilon A(\epsilon^2 t, \epsilon x) \cdot \mathrm{e}^{ix} + c.c. + \mathcal{O}(\epsilon^2)$$

$A(T,X) \in \mathbb{C}$ solves stochastic Ginzburg-Landau equation:

$$\partial_T A = 4\partial_X^2 A + \nu A - 3|A|^2 A + \sigma\eta \qquad \text{(GL)}$$

space-time white noise $\eta(T,X) \in \mathbb{C}$, even if ξ colored in space!

Degenerate Noise on $\mathbb{R}$

$$\partial_t u = \mathcal{L}u + \nu\epsilon^2 u - u^3 + \sigma\epsilon\partial_t\beta \qquad \text{on } \mathbb{R} \qquad \text{(SH)}$$

Ansatz:

$$u(t,x) = \epsilon A(\epsilon^2 t, \epsilon x)e^{ix} + c.c. + \epsilon\sigma Z(t) + \mathcal{O}(\epsilon^2)$$

fast OU-process $Z(t) = \int_0^t e^{-(t-\tau)} d\beta(\tau)$

Result: Amplitude Equation [DB, Mohammed, '11]

$$\partial_T A = 4\partial_X^2 A + (\nu - \tfrac{3}{2}\sigma^2)A - 3A|A|^2 \qquad \text{(GL)}$$

Open Problems – Regularity of A

- Kirrmann-Mielke-Schneider needed $A \in C_b^{1,4}([0,T] \times \mathbb{R})$
- DB-Hairer-Pavliotis needed $A = B + Z$
  with $B \in C^0([0,T], H^1[-L,L])$
  and $Z \in C^0([0,T] \times [-L,L])$ Gaussian
- DB-Mohammed needed $A \in C^0([0,T], H^\alpha(\mathbb{R}))$, $\alpha > 1/2$

But: For space-time white noise
is the Amplitude A only Hölder continuous (Exponent $< \frac{1}{2}$)
and $\|A(T,\cdot)\|_\infty = \infty$ for all $T > 0$.

And: Bounds for terms like

$$e^{t\mathcal{L}}[A(\epsilon x)e^{ix}] \approx [e^{4T\partial_x^2}A](\epsilon x) \cdot e^{ix}$$

require some regularity of A.

Outlook

Further results / Work in Progress

- Higher order corrections – Martingale terms
  [Roberts, Wei Wang, '09], [DB, Mohammed '11]
- Large Domains – Modulated Pattern
  [DB, Klepel, Mohammed]
- Levy noise [DB, Hausenblas]
- Local shape of random invariant manifolds
  [DB, Wei Wang, '09], [Duan, et.al.]
- Approximation of invariant measures [DB, Hairer, '04]

Summary

- SPDEs near a change of stability
- Transient dynamics via amplitude equations
- Stabilisation due to additive noise
- Effect of noise on dominant modes
- Noise transported by nonlinearity between Fourier-modes