



Convergence and Optimality of Adaptive Finite Elements

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1 Variational Problem

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Problem (Weak Formulation of PDEs)

given Hilbert space \mathbb{V} , a continuous bilinear form $\mathcal{B}: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ and $f \in \mathbb{V}^*$ solve

$$u \in \mathbb{V} : \quad \mathcal{B}[u, v] = \langle f, v \rangle \quad \forall v \in \mathbb{V} \quad (\text{VP})$$

Theorem (Nečas, 1962)

*problem (VP) is well-posed in the sense of Hadamard if and only if \mathcal{B} satisfies the **inf-sup condition***

$$\exists c_{\mathcal{B}} > 0 : \quad \inf_{w \in \mathbb{V}} \sup_{v \in \mathbb{V}} \frac{\mathcal{B}[w, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} = \inf_{v \in \mathbb{V}} \sup_{w \in \mathbb{V}} \frac{\mathcal{B}[w, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} = c_{\mathcal{B}} > 0$$



Energy Minimization (Hilbert, 1901)

\mathcal{B} symmetric and coercive (scalar product), i. e.,

$$\exists c_{\mathcal{B}} > 0 : \quad \mathcal{B}[v, v] \geq c_{\mathcal{B}} \|v\|_{\mathbb{V}}^2 \quad \forall v \in \mathbb{V},$$

- (VP) are the **Euler-Lagrange** equations characterizing the unique minimizer u of the energy

$$E(v) = \frac{1}{2} \mathcal{B}[v, v] - \langle f, v \rangle := \frac{1}{2} \|v\|^2 - \langle f, v \rangle$$



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Poisson Problem: $-\Delta u = f$ in Ω , $u = 0$ on $\partial\Omega$

variational form: find $u \in \mathbb{V} = H_0^1(\Omega)$ s.th.

$$\int_{\Omega} \nabla v \cdot \nabla u \, dV = \int_{\Omega} f v \, dV \quad \forall v \in \mathbb{V}$$

- Euler-Lagrange equations for the **Dirichlet-energy**

$$E(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dV - \int_{\Omega} f v \, dV$$



Coercive Bilinear Forms (Lax-Milgram, 1954)

$$\exists c_B > 0 : \quad \mathcal{B}[v, v] \geq c_B \|v\|_{\mathbb{V}}^2 \quad \forall v \in \mathbb{V}$$

- coercivity implies the inf-sup

$$\forall v \in \mathbb{V} : \quad \sup_{w \in \mathbb{V}} \frac{\mathcal{B}[w, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} \geq \frac{\mathcal{B}[v, v]}{\|v\|_{\mathbb{V}}^2} \geq c_B$$



Coercive Bilinear Forms (Lax-Milgram, 1954)

$$\exists c_{\mathcal{B}} > 0 : \quad \mathcal{B}[v, v] \geq c_{\mathcal{B}} \|v\|_{\mathbb{V}}^2 \quad \forall v \in \mathbb{V}$$

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Advection-Diffusion Problem: $-\Delta u + \mathbf{b} \cdot \nabla u = f$ in Ω , $u = 0$ on $\partial\Omega$

variational form: find $u \in \mathbb{V} = H_0^1(\Omega)$ s.th.

$$\int_{\Omega} \nabla v \cdot \nabla u + v \mathbf{b} \cdot \nabla u \, dV = \int_{\Omega} f v \, dV \quad \forall v \in \mathbb{V}$$

- \mathcal{B} is coercive for $\operatorname{div} \mathbf{b} = 0$
- inf-sup of \mathcal{B} for general \mathbf{b}



Constrained Minimization (Brezzi, 1974)

assume \mathcal{B} is a scalar product and minimize $E(v)$ over a closed, non-empty subspace $\mathbb{V}_0 \subset \mathbb{V}$.



Special Cases: Saddle-Point Problems

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Constrained Minimization (Brezzi, 1974)

assume \mathcal{B} is a scalar product and minimize $E(v)$ over a closed, non-empty subspace $\mathbb{V}_0 \subset \mathbb{V}$.

Stokes System: $-\Delta \mathbf{u} + \nabla p = \mathbf{f}$ in Ω , $\operatorname{div} \mathbf{u} = 0$ in Ω , $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$

minimize the Dirichlet-energy over

$$\mathbb{V}_0 = \{ \mathbf{v} \in \mathbb{V} \mid \operatorname{div} \mathbf{v} = 0 \} \subset \mathbb{V} = H_0^1(\Omega; \mathbb{R}^d)$$

the stationary point $(\mathbf{u}, p) \in \mathbb{W} = H_0^1(\Omega; \mathbb{R}^d) \times \{q \in L_2(\Omega) \mid \int_{\Omega} q \, dV = 0\}$ of the associated Lagrangian satisfies

$$\int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{u} \, dx - \int_{\Omega} \operatorname{div} \mathbf{v} \, p \, dx - \int_{\Omega} \operatorname{div} \mathbf{u} \, q \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx$$

for all (\mathbf{v}, q)

- inf-sup equivalent to solvability of the divergence equation
- p is the Lagrange multiplier from constrained energy minimization



Discretization: Galerkin Approximation/Ritz Procedure

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Discrete Problem

for a subspace $\mathbb{V}_N \subset \mathbb{V}$ of dimension $N < \infty$ solve

$$U_N \in \mathbb{V}_N : \quad \mathcal{B}[U_N, V] = \langle f, V \rangle \quad \forall V \in \mathbb{V}_N \quad (\text{DVP})$$

Theorem

problem (DVP) is well-posed iff \mathcal{B} satisfies the discrete inf-sup condition

$$\exists c_N > 0 : \quad \inf_{W \in \mathbb{V}_N} \sup_{V \in \mathbb{V}_N} \frac{\mathcal{B}[W, V]}{\|W\|_{\mathbb{V}} \|V\|_{\mathbb{V}}} \geq c_N$$



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$$\exists c_N > 0 : \quad \inf_{W \in \mathbb{V}_N} \sup_{V \in \mathbb{V}_N} \frac{\mathcal{B}[W, V]}{\|W\|_{\mathbb{V}} \|V\|_{\mathbb{V}}} \geq c_N$$

Remarks

- second inf-sup follows from $\dim \mathbb{V}_N = N < \infty$
- for coercive \mathcal{B} we have $c_N \geq c_{\mathcal{B}}$
- Ritz procedure for symmetric and coercive \mathcal{B} , 1909:

$$U_N = \arg \min_{V \in \mathbb{V}_N} E(V)$$

- Bubnov & Galerkin generalization to non-symmetric \mathcal{B} , 1915



Quasi-Best Approximation Property

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Theorem (Babuška, 1971; Xu, Zikatanov, 2003)

the discrete solution is a quasi-optimal choice from \mathbb{V}_N :

$$\|U_N - u\|_V \leq \frac{\|\mathcal{B}\|}{c_N} \min_{V \in \mathbb{V}_N} \|V - u\|_V$$



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$$\|U_N - u\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{c_N} \min_{V \in \mathbb{V}_N} \|V - u\|_{\mathbb{V}}$$

Remarks

- for coercive \mathcal{B} this is Cea's Lemma from 1964

$$\|U_N - u\|_{\mathbb{V}} \leq \frac{\|\mathcal{B}\|}{c_{\mathcal{B}}} \min_{V \in \mathbb{V}_N} \|V - u\|_{\mathbb{V}},$$

since $c_N \geq c_{\mathcal{B}}$

- best approximation if \mathcal{B} is a scalar product

$$\|U_N - u\| = \min_{V \in \mathbb{V}_N} \|V - u\|$$

- in case of non-coercive \mathcal{B} , stable discretizations with $c_N \geq \underline{c} > 0$ are important for optimal estimates



quasi-optimality of the Galerkin approximation

$$\|U_N - u\|_V \leq \frac{\|\mathcal{B}\|}{c_N} \min_{V \in \mathbb{V}_N} \|V - u\|_V$$

has the simple consequence

Corollary

if $c_N \geq \underline{c} > 0$ then

$$\lim_{N \rightarrow \infty} \|U_N - u\|_V = 0 \quad \iff \quad \lim_{N \rightarrow \infty} \text{dist}(u, \mathbb{V}_N) = 0$$



Uniform vs. Adaptive Approximation

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uniform approximation aims at

$$\text{dist}(v, \mathbb{V}_N) \rightarrow 0 \quad \text{for any } v \in \mathbb{V}$$

Remarks

- plain convergence is a consequence of density

$$\mathbb{V} = \overline{\bigcup_{N \geq 0} \mathbb{V}_N}^{\|\cdot\|_{\mathbb{V}}}$$

- plain convergence does not require any additional regularity of u
- a convergence rate is derived from standard interpolation estimates (e. g., in terms of maximal mesh size)
 - any rate requires additional regularity of u above \mathbb{V}



adaptive approximation only aims at

$$\text{dist}(u, \mathbb{V}_N) \rightarrow 0 \quad \text{for the solution } u$$

Remarks

- a necessary and sufficient condition for convergence is

$$u \in \mathbb{V}_\infty := \overline{\bigcup_{N \geq 0} \mathbb{V}_N}^{\|\cdot\|_{\mathbb{V}}};$$

in general $\mathbb{V}_\infty \neq \mathbb{V}$

- criteria for the construction of \mathbb{V}_N non obvious; should not require additional regularity of u
- convergence rate only meaningful in terms of dimension of \mathbb{V}_N and only optimal for a good choice of spaces \mathbb{V}_N
 - any rate requires additional regularity of u above \mathbb{V}



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Piecewise Constant Approximation

approximate $u \in C^0([0, 1])$ by a piecewise constant function U_N on a partition

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

- for instance,

$$U_N(x) = u(x_n) \quad \text{for } x \in I_n := (x_{n-1}, x_n], \quad n = 1, \dots, N$$



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- for instance,

$$U_N(x) = u(x_n) \quad \text{for } x \in I_n := (x_{n-1}, x_n], \quad n = 1, \dots, N$$

Convergence

uniform continuity of u implies

$$\lim_{N \rightarrow \infty} \|U_N - u\|_{L^\infty(0,1)} = 0$$

provided that

$$h_N := \max_{n=1, \dots, N} |x_n - x_{n-1}| \rightarrow 0$$



Uniform Approximation

choose x_n as

$$x_n = n/N, \quad n = 1, \dots, N$$

- error bound for $u \in W_\infty^1(0, 1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \leq h_N \|u'\|_{L_\infty(0,1)} = N^{-1} \|u'\|_{L_\infty(0,1)}$$

Uniform Approximation

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Adaptive Approximation

choose x_n s.th.

$$\int_{x_{n-1}}^{x_n} |u'(s)| ds = \frac{1}{N} \|u'\|_{L_1(0,1)}, \quad n = 1, \dots, N$$

- error bound for $u \in W_1^1(0, 1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \leq N^{-1} \|u'\|_{L_1(0,1)}$$



Uniform Approximation

- quantification of the rate in terms of mesh-size and number of degrees of freedom is equivalent: $h_N = N^{-1}$

Adaptive Approximation

- quantification of the rate in terms of the mesh-size is impossible, only in terms of degrees of freedom; in general, $h_N \not\rightarrow 0$



Uniform Approximation

- quantification of the rate in terms of mesh-size and number of degrees of freedom is equivalent: $h_N = N^{-1}$
- choice of the grid points is independent of u and can be used for any function

Adaptive Approximation

- quantification of the rate in terms of the mesh-size is impossible, only in terms of degrees of freedom; in general, $h_N \not\rightarrow 0$
- choice of the grid points depends on u is cannot be used for another function (**Nonlinear Approximation**)



Uniform Approximation

- quantification of the rate in terms of mesh-size and number of degrees of freedom is equivalent: $h_N = N^{-1}$
- choice of the grid points is independent of u and can be used for any function
- error bound needs stronger regularity $u \in W_\infty^1(0, 1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \leq N^{-1} \|u'\|_{L_\infty(0,1)}$$

Adaptive Approximation

- quantification of the rate in terms of the mesh-size is impossible, only in terms of degrees of freedom; in general, $h_N \not\rightarrow 0$
- choice of the grid points depends on u is cannot be used for another function (**Nonlinear Approximation**)
- error bound needs less regularity $u \in W_1^1(0, 1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \leq N^{-1} \|u'\|_{L_1(0,1)}$$



Problem

given $u \in C^0([0, 1])$ and $\text{TOL} > 0$ choose N minimal s.th.

$$\|U_N - u\|_{L_\infty(0,1)} \leq \text{TOL}$$

- task accomplished with minimal N s.th.

$$N_{\text{uniform}}^{-1} \|u'\|_{L_\infty(0,1)} \leq \text{TOL} \quad \text{resp.} \quad N_{\text{adaptive}}^{-1} \|u'\|_{L_1(0,1)} \leq \text{TOL}$$

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compare uniform and adaptive approximation for

$$u(x) := x^\alpha$$

in case of

$$\alpha \geq 1: u \in W_\infty^1(0, 1) \text{ (regular)}$$

$$0 < \alpha < 1: u \notin W_\infty^1(0, 1) \text{ (singular)} \text{ but } u \in W_1^1(0, 1)$$



Comparison for a Regular Solution

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$\alpha \geq 1$:

$$\|u'\|_{L^\infty(0,1)} = \max_{x \in [0,1]} \alpha x^{\alpha-1} = \alpha, \quad \|u'\|_{L_1(0,1)} = \int_0^1 \alpha x^{\alpha-1} dx = 1$$

- tolerances and number of nodes

$$N_{\text{uniform}} = \lceil \alpha \text{TOL}^{-1} \rceil \quad \text{resp.} \quad N_{\text{adaptive}} = \lceil \text{TOL}^{-1} \rceil$$

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Comparison for $\alpha = 10$

TOL	N_{uniform}	N_{adaptive}
1%	10^3	10^2
0.01%	10^5	10^4

- adaptive approximation more efficient but only by the factor $\alpha \geq 1$

$\alpha < 1$: for uniform refinement we have

$$\|U_N - u\|_{L_\infty(0,1)} = |U_N(x_0) - u(x_0)| = |u(x_1)| = N^{-\alpha}$$

- tolerances and number of nodes

$$N_{\text{uniform}} = \lceil \text{TOL}^{-1/\alpha} \rceil \quad \text{resp.} \quad N_{\text{adaptive}} = \lceil \text{TOL}^{-1} \rceil$$

Comparison for a Singular Solution

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Comparison for $\alpha = 0.1$

TOL	N_{uniform}	N_{adaptive}
1%	10^{20}	10^2
0.01%	10^{40}	10^4

- adaptive approximation independent of α !

Comparison for a Singular Solution

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- uniform approximation with tolerance 1% on **Hermit** (1 PetaFlop) in Stuttgart will need more than 27h

Comparison for a Singular Solution

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Comparison for $\alpha = 0.1$

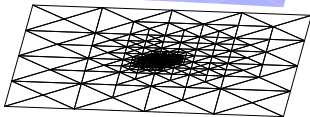
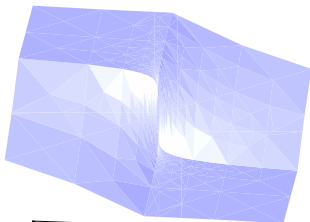
TOL	N_{uniform}	N_{adaptive}
1%	10^{20}	10^2
0.01%	10^{40}	10^4

- adaptive approximation independent of α !
- uniform approximation with tolerance 1% on **Hermit** (1 PetaFlop) in Stuttgart will need more than 27h
- tolerance 0.01% trivial with adaptive approximation (< 1 s on this laptop) but not feasible on any computer with uniform approximation

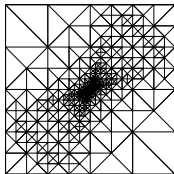


typical singularities of solutions to PDEs are of the form $|x|^\alpha$, $\alpha \in (0, 1)$

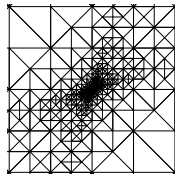
jumping coefficients



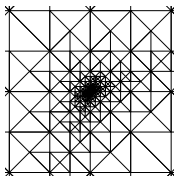
singularity $\approx |x|^{0.1}$



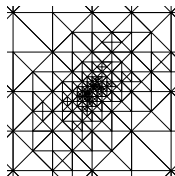
grid < 2000 nodes



zoom: $(-10^{-3}, 10^{-3})^2$



zoom: $(-10^{-6}, 10^{-6})^2$



zoom: $(-10^{-9}, 10^{-9})^2$

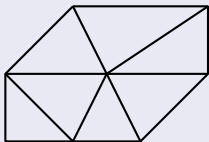


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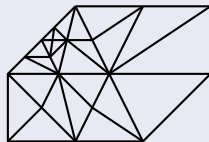


Adaptive Finite Element Discretization

- start with a coarse, conforming triangulation \mathcal{G}_0 of Ω
- let \mathbb{G} be the set of all conforming refinements of \mathcal{G}_0 created by bisection



\mathcal{G}_0



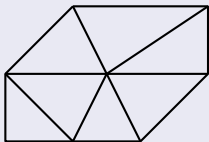
$\mathcal{G} \in \mathbb{G}$

- let $\mathbb{V}(\mathcal{G}) \subset \mathbb{V}$ be a piecewise polynomial finite element space over \mathcal{G} .

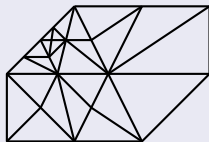


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\mathcal{G}_0



$\mathcal{G} \in \mathbb{G}$

- let $\mathbb{V}(\mathcal{G}) \subset \mathbb{V}$ be a piecewise polynomial finite element space over \mathcal{G} .

bisection refinement of simplicial grids and polynomials is not essential but

Nesting of Spaces

if \mathcal{G}_+ is a refinement of \mathcal{G} the finite element spaces are nested:

$$\mathcal{G} \leq \mathcal{G}_+ \quad \implies \quad \mathbb{V}(\mathcal{G}) \subset \mathbb{V}(\mathcal{G}_+)$$



Aim of Adaptive Methods

given $\text{TOL} > 0$ find $\mathcal{G} \in \mathbb{G}$ s.th.

- $\|U_{\mathcal{G}} - u\|_{\mathbb{V}} \leq \text{TOL}$
- $\#\mathcal{G}$ is minimal

constrained, discrete minimization problem ...



Aim of Adaptive Methods

given $\text{TOL} > 0$ find $\mathcal{G} \in \mathbb{G}$ s.th.

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- $\#\mathcal{G}$ is minimal

constrained, discrete minimization problem ...

too hard to solve ...

Equidistribution Principle (Babuška, Rheinboldt, 1978)

optimal grid \mathcal{G} satisfies

$$\|U_{\mathcal{G}} - u\|_{\mathbb{V}(E)} \equiv \text{const.} \quad \forall E \in \mathcal{G}$$



Problems

- true solution u not known \implies true error not known
- 1d error equidistribution does not extend to higher dimensions



Problems

- true solution u not known \implies true error not known
- 1d error equidistribution does not extend to higher dimensions

Adaptive Approaches

- replace the true error by an a posteriori estimator, i. e., a computable bound for the true error in terms of the discrete solution and data
- iteratively perform local refinement/coarsening of selected elements based on information of the estimator
- selection of elements subject for refinement/coarsening aims at equidistribution of the indicators



Standard Adaptive Iteration

iterate

SOLVE \longrightarrow ESTIMATE \longrightarrow MARK \longrightarrow REFINE

to generate a sequence $\{\mathcal{G}_k, U_k\}_{k \geq 0}$ of grids and discrete solutions

- **SOLVE:** computes the Galerkin approximation $U_k \in \mathbb{V}_k = \mathbb{V}(\mathcal{G}_k)$
 - i. e., exact integration and exact linear algebra
- **ESTIMATE:** computes error indicators $\{\mathcal{E}_k(U_k, E)\}_{E \in \mathcal{G}_k}$
- **MARK:** selects elements in \mathcal{G}_k for refinement
- **REFINE:** refines all marked elements and outputs a new grid



Definition (Residual)

the residual $\mathcal{R}(U_G) \in \mathbb{V}^*$ is an **a posteriori quantity** defined by

$$\langle \mathcal{R}(U_G), v \rangle := \mathcal{B}[U_G, v] - \langle f, v \rangle \quad \forall v \in \mathbb{V}$$



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Error and Residual

equivalence of $\|U_G - u\|_{\mathbb{V}}$ to the **non-computable** quantity $\|\mathcal{R}(U_G)\|_{\mathbb{V}^*}$

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Error Estimator

an estimator is a **computable bound** for the residual with

$$\begin{aligned} \|\mathcal{R}(U_G)\|_{\mathbb{V}^*} &\lesssim \mathcal{E}_{\mathcal{G}}(U_G; \mathcal{G}) \\ \mathcal{E}_{\mathcal{G}}(U_G; E) &\lesssim \|\mathcal{R}(U_G)\|_{\mathbb{V}^*(\omega_E)} + \text{osc}_{\mathcal{G}}(U_G; \omega_E), \quad E \in \mathcal{G} \end{aligned}$$

- constants solely depend on the shape regularity of \mathcal{G}



Properties of an Estimator: Reliability and Efficiency

there are constants C_1, C_2 s.th. the Galerkin approximation U_G satisfies

$$C_1 \|U_G - u\|_{\mathbb{V}}^2 \leq \mathcal{E}_G^2(U_G; \mathcal{G}) := \sum_{E \in \mathcal{G}} \mathcal{E}_G^2(U_G; E)$$

$$\mathcal{E}_G^2(U_G; E) \leq C_2 \|U_G - u\|_{\mathbb{V}(\omega_E)}^2 + \text{osc}_G^2(U_G; \omega_E), \quad E \in \mathcal{G}$$



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Example

- the indicators of the residual estimator for the Poisson problem read

$$\mathcal{E}_G^2(U_G; E) := \|h_G (-\Delta U_G - f)\|_{L_2(E)}^2 + \|h_G^{1/2} \llbracket \nabla U_G \rrbracket\|_{L_2(\partial E \cap \Omega)}^2,$$

where $h_G \in L_\infty(\Omega)$ is the piecewise constant mesh-size function with

$$h_{G|E} = |E|^{1/d} \approx \text{diam}(E), \quad E \in \mathcal{G}$$

- oscillation is here data oscillation and given by

$$\text{osc}_G^2(U_G; E) := \|h_G(f_G - f)\|_{L_2(E)}^2$$



marking strategies select elements for refinement based on the indicators $\{\mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}; E)\}_{E \in \mathcal{G}}$; typical marking strategies are

Equidistribution Strategy

Parameter $\theta \in [0, 1]$

$$\mathcal{E}_* = \theta \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}; \mathcal{G}) / \#\mathcal{G}^{1/2}$$

$$\mathcal{M} = \{E \in \mathcal{G} \mid \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}; E) \geq \mathcal{E}_*\}$$

Maximum Strategy

Parameter $\nu \in [0, 1]$

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Dörfler Marking

Parameter $\theta \in (0, 1]$

select any $\mathcal{M} \subset \mathcal{G}$ s.th.

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- Dörfler Marking is also known as **Fixed Fraction Marking**, **Bulk Criterion**, **Guaranteed Error Reduction Strategy**



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Minimal Dörfler Marking

Parameter $\theta \in (0, 1]$

select **minimal** $\mathcal{M} \subset \mathcal{G}$ s.th.

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Basic Questions

- 1 does the sequence of adaptively generated solutions converge, i. e.,

$$\lim_{k \rightarrow \infty} \|U_k - u\|_{\mathbb{V}} = 0?$$

- 2 if yes, does the sequence converge with a rate, i. e.,

$$\|U_k - u\|_{\mathbb{V}} \lesssim |u|_s (\#\mathcal{G}_k - \#\mathcal{G}_0)^{-s}?$$



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Problems in the Analysis

- 1 all decision are taken at step k , no information about U_ℓ and \mathcal{G}_ℓ available for $\ell > k$
- 2 for scalar products \mathcal{B} the quantity $\|U_k - u\|$ is monotone but not strictly monotone
- 3 the estimator \mathcal{E}_k is not monotone, no relation between estimators of different iterations
- 4 U_k is a global object: refinement of a single element affects U_{k+1} everywhere; no quantification of these effects available



- 1 Variational Problem
- 2 Uniform and Adaptive Approximation
- 3 Adaptive Finite Element Methods
- 4 Convergence
- 5 Convergence Rates



Theorem (Convergence)

the adaptively generated sequence $\{U_k\}_{k \geq 0}$ converges in \mathbb{V} to the solution u for the general class of inf-sup problems, provided

- 1** *the sequence of discrete spaces is nested and uniformly inf-sup stable;*
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Main Ideas of the Proof

- 1** **uniform** convergence of $\{h_k\}_{k \geq 0}$ to some $h_\infty \in L_\infty(\Omega)$
 - $h_\infty \neq 0$ in general



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1 **uniform** convergence of $\{h_k\}_{k \geq 0}$ to some $h_\infty \in L_\infty(\Omega)$

- $h_\infty \neq 0$ in general

2 strong convergence of $\{U_k\}_{k \geq 0}$ to the solution $u_\infty \in \mathbb{V}_\infty$ of the variational problem in

$$\mathbb{V}_\infty = \overline{\bigcup_{k \geq 0} \mathbb{V}_k}^{\|\cdot\|_{\mathbb{V}}} \subset \mathbb{V}$$

- $\mathbb{V}_\infty = \mathbb{V}$ iff $h_\infty \equiv 0$



Main Ideas of the Proof

3 convergence $U_k \rightarrow u_\infty$, $h_k \rightarrow h_\infty$, properties of \mathcal{E}_k and MARK yield

$$\max\{\mathcal{E}_k(U_k; E) \mid E \in \mathcal{G}_k\} \rightarrow 0$$



Main Ideas of the Proof

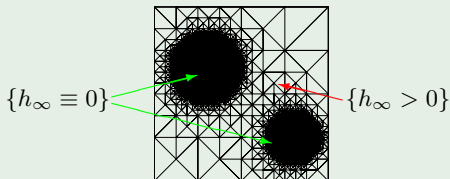
3 convergence $U_k \rightarrow u_\infty$, $h_k \rightarrow h_\infty$, properties of \mathcal{E}_k and MARK yield

$$\max\{\mathcal{E}_k(U_k; E) \mid E \in \mathcal{G}_k\} \rightarrow 0$$

4 deduce $\mathcal{R}(u_\infty) = 0$ from

$$|\langle \mathcal{R}(U_k), v \rangle| = |\langle \mathcal{R}(U_k), v - I_k v \rangle| \lesssim \sum_{E \in \mathcal{G}_k} \mathcal{E}_k(U_k; E) \|v - I_k v\|_{\mathbb{V}(E)} \rightarrow 0$$

for **smooth** v by employing local density of the FE spaces in $\{h_\infty \equiv 0\}$, and convergence of the indicators in its complement $\{h_\infty > 0\}$. \square





Remarks

plain convergence of adaptive finite elements is more or less settled

- 1 basic tool for a convergence result for the heat equation (Kreuzer, Möller, Schmidt, S. 2012)
- 2 transfer to control constrained optimal control problems (Kohls, Rösch, S. 2012)

generalizations seem to work for

- 1 other non-linear problems
 - needs strong convergence

$$\|U_k - u_\infty\|_V \rightarrow 0$$

(for instance in strictly convex minimization problems)

- 2 non-nested spaces
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Drawback

arguments give no error reduction property, which is currently an essential tool to derive a convergence rate

- cannot be expected for this general problem class



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Quality of Best-Approximation

let $\mathbb{G}_N := \{\mathcal{G} \in \mathbb{G} \mid \#\mathcal{G} - \#\mathcal{G}_0 \leq N\}$ and set

$$\sigma(N; u) := \min_{\mathcal{G} \in \mathbb{G}_N} \min_{V \in \mathcal{V}(\mathcal{G})} \|u - V\|$$



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Approximation Class

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Remarks

$u \in \mathcal{A}_s$ implies the existence of a sequence $\{\mathcal{G}_k\}_{k \geq 0}$ s.th.

$$\min_{V_k \in \mathbb{V}(\mathcal{G}_k)} \|V_k - u\| \lesssim |u|_s (\#\mathcal{G}_k - \#\mathcal{G}_0)^{-s} \rightarrow 0$$

- this is **not** the sequence produced by the adaptive method
- any rate $s > 0$ requires additional regularity of u above \mathbb{V} , e. g. H^1



Lemma (Constructive Approximation)

if $\mathbb{V} = H_0^1(\Omega)$, $d = 2$, and $u \in W_p^2(\Omega)$ for some $p > 1$ then $u \in \mathcal{A}_{1/2}$



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Remarks

- 1 constructive proof relying on
 - Lagrange interpolant $I_{\mathcal{G}}u$ being completely local
 - the strictly monotone and local error bound

$$\|I_{\mathcal{G}}u - u\|_E = \|\nabla(I_{\mathcal{G}}u - u)\|_{H^1(E)} \lesssim h_E^{2-2/p} \|D^2u\|_{p;E}$$

- 2 proof based on Binev, Dahmen, DeVore, and Petrushev, who give a near characterization of \mathcal{A}_s in terms of Besov spaces for Courant elements, 2002
- 3 rate only optimal for Courant elements in 2d
- 4 higher order elements and $d \geq 3$ need non-convex spaces ($p < 1$); compare with Gaspoz and Morin, 2011
- 5 proof requires a complexity result for refinement by bisection . . .



Theorem (Complexity of Refinement by Bisection)

for a compatible labeling of refinement edges on \mathcal{G}_0 we have

$$\#\mathcal{G}_{k+1} - \#\mathcal{G}_0 \lesssim \sum_{\ell=0}^k \#\mathcal{M}_\ell \quad \forall k \geq 0,$$

i. e., the total number of generated elements is bounded by the total number of marked elements



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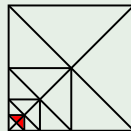
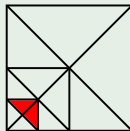
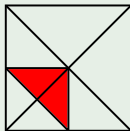
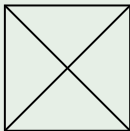
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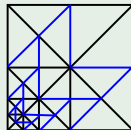
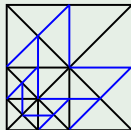
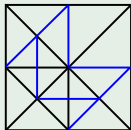
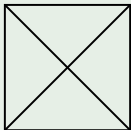
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Assumptions on AFEM

- 1 Poisson problem
- 2 the standard residual estimator $\mathcal{E}_{\mathcal{G}}$
- 3 *minimal* Dörfler Marking with $0 < \theta^2 < \theta_*^2 \approx C_2/C_1$

$$\theta \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}, \mathcal{G}) \leq \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}, \mathcal{M})$$

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Basic Property from Minimizing an Energy

for Ritz-projections $U \in \mathbb{V}(\mathcal{G})$ and $U_+ \in \mathbb{V}(\mathcal{G}_+)$ with $\mathbb{V}(\mathcal{G}) \subset \mathbb{V}(\mathcal{G}_+)$ we have the orthogonal error decomposition

$$\|U_+ - u\|^2 = \|U - u\|^2 - \|U - U_+\|^2$$



Proposition (Contraction Property)

there are constants $\gamma > 0$ and $0 < \alpha < 1$ s.th.

$$\|U_{k+1} - u\|^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}; \mathcal{G}_{k+1}) \leq \alpha \left(\|U_k - u\|^2 + \gamma \mathcal{E}_k^2(U_k; \mathcal{G}_k) \right)$$



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Philosophy behind the Proof

combine the following two extreme cases:

- 1 $U_{k+1} = U_k$: then $\mathcal{E}_{k+1}^2(U_{k+1}; \mathcal{G}_{k+1}) \leq \bar{\alpha} \mathcal{E}_k^2(U_k; \mathcal{G}_k)$ by the strict reduction of the local mesh-size and Dörfler Marking

$$\mathcal{E}_k^2(U_k; E) = \|h_k (-\Delta U_k - f)\|_{L_2(E)}^2 + \|h_k^{1/2} [\nabla U_k]\|_{L_2(\partial E \cap \Omega)}^2$$

- this case corresponds to large oscillation
- 2 zero oscillation: then $\|U_{k+1} - u\|^2 \leq \bar{\alpha} \|U_k - u\|^2$ from the orthogonal error decomposition and Dörfler Marking
 - compensates even for a possible increase of $\mathcal{E}_k(U_k; \mathcal{G})$ □



Theorem (Optimal Rate of AFEM)

if $u \in \mathcal{A}_s$ then the sequence of adaptively generated solutions U_k satisfies

$$\|U_k - u\| + \mathcal{E}_k(U_k; \mathcal{G}_k) \lesssim |u|_s (\#\mathcal{G}_k - \#\mathcal{G}_0)^{-s}$$



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Basic Steps of the Proof

- 1 localized upper bound for two Ritz approximations: a suitable error reduction yields the Dörfler property for the set of refined elements
- 2 minimal Dörfler Marking gives a connection of the adaptive mesh \mathcal{G}_k with an optimal one
- 3 the connection to an optimal mesh yields an upper bound of $\#\mathcal{M}_k$ in terms of the actual error
- 4 proposition follows from complexity of REFINE and the contraction property





Generalizations

- 1 extension to general scalar 2^{nd} order, linear, symmetric PDEs and higher order elements (Cascón, Kreuzer, Nochetto, S. 2008)
 - non-standard approximation class, depends also on oscillation
- 2 extension to other estimators being locally equivalent to the residual estimator (Kreuzer, S. 2011; Cascón, Nochetto 2012)



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Open Problems

- 1 other marking strategies are excluded
- 2 proof heavily relies on the orthogonal error decomposition; extension to non-symmetric and non-coercive \mathcal{B} completely open
- 3 lack of a localized upper bound for two Ritz projections in case of $H(\text{curl})$ or $H(\text{div})$ problems
 - essential ingredient in the proof
- 4 ...



Directly Related

- 1 Dörfler
- 2 Binev, Dahmen, DeVore
- 3 Cascón, Kreuzer, Nochetto, S.
- 4 Kreuzer, S.
- 5 Morin, Nochetto, S.
- 6 Morin, S., Veerer
- 7 [Nochetto, S., Veerer](#)
- 8 S.
- 9 Stevenson

Background

- 1 Bänsch, Morin, Nochetto
- 2 Carstensen
- 3 Cascón, Nochetto, S.
- 4 Chen, Feng
- 5 Chen, Holst, Xu
- 6 Carstensen, Hoppe
- 7 Diening, Kreuzer
- 8 Mekchay, Nochetto
- 9 Veerer
- 10 Veerer, S.
- 11 ...

Thank you for your interest!