Convergence and Optimality of Adaptive Finite Elements

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Contents

Outline

PDE given as Variational Problem

Problem (Weak Formulation of PDEs)

given Hilbert space $\mathbb{V}$, a continuous bilinear form $\mathcal{B}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ and $f \in \mathbb{V}^*$ solve

$$u \in \mathbb{V} : \qquad \mathcal{B}[u, v] = \langle f, v \rangle \qquad \forall v \in \mathbb{V} \tag{VP}$$

Theorem (Nečas, 1962)

problem (VP) *is well-posed in the sense of Hadamard if and only if* $\mathcal{B}$ *satisfies the inf-sup condition*

$$\exists c_{\mathcal{B}} > 0 : \qquad \inf_{w \in \mathbb{V}} \sup_{v \in \mathbb{V}} \frac{\mathcal{B}[w, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} = \inf_{v \in \mathbb{V}} \sup_{w \in \mathbb{V}} \frac{\mathcal{B}[w, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} = c_{\mathcal{B}} > 0$$

Special Cases: Symmetric and Coercive Bilinear Forms

Energy Minimization (Hilbert, 1901)

$\mathcal{B}$ symmetric and coercive (scalar product), i. e.,

$$\exists c_{\mathcal{B}} > 0 : \qquad \mathcal{B}[v,\, v] \geq c_{\mathcal{B}} \|v\|_{\mathbb{V}}^2 \qquad \forall v \in \mathbb{V},$$

- (VP) are the Euler-Lagrange equations characterizing the unique minimizer u of the energy

$$E(v) = \frac{1}{2}\mathcal{B}[v,\, v] - \langle f,\, v\rangle := \frac{1}{2} |||v|||^2 - \langle f,\, v\rangle$$

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Poisson Problem: $-\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$

variational form: find $u \in \mathbb{V} = H_0^1(\Omega)$ s.th.

$$\int_\Omega \nabla v \cdot \nabla u \, dV = \int_\Omega f v \, dV \qquad \forall\, v \in \mathbb{V}$$

- Euler-Lagrange equations for the Dirichlet-energy

$$E(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 \, dV - \int_\Omega f v \, dV$$

Special Cases: Coercive Problems

Coercive Bilinear Forms (Lax-Milgram, 1954)

$$\exists c_{\mathcal{B}} > 0 : \qquad \mathcal{B}[v,\, v] \geq c_{\mathcal{B}} \|v\|_{\mathbb{V}}^2 \qquad \forall v \in \mathbb{V}$$

- coercivity implies the inf-sup

$$\forall v \in \mathbb{V} : \qquad \sup_{w \in \mathbb{V}} \frac{\mathcal{B}[w,\, v]}{\|w\|_{\mathbb{V}} \|v\|_{\mathbb{V}}} \geq \frac{\mathcal{B}[v,\, v]}{\|v\|_{\mathbb{V}}^2} \geq c_{\mathcal{B}}$$

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Advection-Diffusion Problem: $-\Delta u + \boldsymbol{b} \cdot \nabla u = f$ in Ω, $u = 0$ on $\partial\Omega$

variational form: find $u \in \mathbb{V} = H_0^1(\Omega)$ s.th.

$$\int_\Omega \nabla v \cdot \nabla u + v \boldsymbol{b} \cdot \nabla u \, dV = \int_\Omega f v \, dV \qquad \forall\, v \in \mathbb{V}$$

- $\mathcal{B}$ is coercive for $\operatorname{div} \boldsymbol{b} = 0$
- inf-sup of $\mathcal{B}$ for general $\boldsymbol{b}$

Special Cases: Saddle-Point Problems

Constrained Minimization (Brezzi, 1974)

assume $\mathcal{B}$ is a scalar product and minimize $E(v)$ over a closed, non-empty subspace $\mathbb{V}_0 \subset \mathbb{V}$.

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Stokes System: $-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$ in Ω, $\operatorname{div} \boldsymbol{u} = 0$ in Ω, $\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$

minimize the Dirichlet-energy over

$$\mathbb{V}_0 = \big\{\boldsymbol{v} \in \mathbb{V} \mid \operatorname{div} \boldsymbol{v} = 0\big\} \subset \mathbb{V} = H_0^1(\Omega; \mathbb{R}^d)$$

the stationary point $(\boldsymbol{u}, p) \in \mathbb{W} = H_0^1(\Omega; \mathbb{R}^d) \times \{q \in L_2(\Omega) \mid \int_\Omega q \, dV = 0\}$ of the associated Lagrangian satisfies

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{u} \, dx - \int_\Omega \operatorname{div} \boldsymbol{v} \, p \, dx - \int_\Omega \operatorname{div} \boldsymbol{u} \, q \, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx$$

for all $(\boldsymbol{v}, q)$

- inf-sup equivalent to solvability of the divergence equation
- p is the Lagrange multiplier from constrained energy minimization

Discretization: Galerkin Approximation/Ritz Procedure

Discrete Problem

for a subspace $\mathbb{V}_N \subset \mathbb{V}$ of dimension $N < \infty$ solve

$$U_N \in \mathbb{V}_N : \qquad \mathcal{B}[U_N, V] = \langle f, V \rangle \qquad \forall\, V \in \mathbb{V}_N \tag{DVP}$$

Theorem

problem (DVP) *is well-posed iff $\mathcal{B}$ satisfies the discrete inf-sup condition*

$$\exists c_N > 0 : \qquad \inf_{W \in \mathbb{V}_N} \sup_{V \in \mathbb{V}_N} \frac{\mathcal{B}[W, V]}{\|W\|_{\mathbb{V}} \|V\|_{\mathbb{V}}} \geq c_N$$

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Remarks

- second inf-sup follows from $\dim \mathbb{V}_N = N < \infty$
- for coercive $\mathcal{B}$ we have $c_N \geq c_{\mathcal{B}}$
- Ritz procedure for symmetric and coercive $\mathcal{B}$, 1909:

$$U_N = \arg\min_{V \in \mathbb{V}_N} E(V)$$

- Bubnov & Galerkin generalization to non-symmetric $\mathcal{B}$, 1915

Quasi-Best Approximation Property

Theorem (Babuška, 1971; Xu, Zikatanov, 2003)

the discrete solution is a quasi-optimal choice from $\mathbb{V}_N$*:*

$$\|U_N - u\|_{\mathbb{V}} \le \frac{\|\mathcal{B}\|}{c_N} \min_{V \in \mathbb{V}_N} \|V - u\|_{\mathbb{V}}$$

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Remarks

- for coercive $\mathcal{B}$ this is Cea's Lemma from 1964

  $$\|U_N - u\|_{\mathbb{V}} \le \frac{\|\mathcal{B}\|}{c_{\mathcal{B}}} \min_{V \in \mathbb{V}_N} \|V - u\|_{\mathbb{V}},$$

  since $c_N \ge c_{\mathcal{B}}$
- best approximation if $\mathcal{B}$ is a scalar product

  $$\|U_N - u\| = \min_{V \in \mathbb{V}_N} \|V - u\|$$

- in case of non-coercive $\mathcal{B}$, stable discretizations with $c_N \ge \underline{c} > 0$ are important for optimal estimates

Characterization of Convergence

quasi-optimality of the Galerkin approximation

$$\|U_N - u\|_{\mathbb{V}} \le \frac{\|\mathcal{B}\|}{c_N} \min_{V \in \mathbb{V}_N} \|V - u\|_{\mathbb{V}}$$

has the simple consequence

Corollary

if $c_N \ge \underline{c} > 0$ then

$$\lim_{N\to\infty} \|U_N - u\|_{\mathbb{V}} = 0 \qquad \Longleftrightarrow \qquad \lim_{N\to\infty} \operatorname{dist}(u, \mathbb{V}_N) = 0$$

Uniform vs. Adaptive Approximation

uniform approximation aims at

$$\operatorname{dist}(v, \mathbb{V}_N) \to 0 \qquad \text{for any } v \in \mathbb{V}$$

Remarks

- plain convergence is a consequence of density

$$\mathbb{V} = \overline{\bigcup_{N \geq 0} \mathbb{V}_N}^{\|\cdot\|_{\mathbb{V}}}$$

  - plain convergence does not require any additional regularity of u
- a convergence rate is derived from standard interpolation estimates (e. g., in terms of maximal mesh size)
  - any rate requires additional regularity of u above $\mathbb{V}$

Uniform vs. Adaptive Approximation

adaptive approximation only aims at

$$\operatorname{dist}(u, \mathbb{V}_N) \to 0 \qquad \text{for the solution } u$$

Remarks

- a necessary and sufficient condition for convergence is

  $$u \in \mathbb{V}_\infty := \overline{\bigcup_{N \geq 0} \mathbb{V}_N}^{\|\cdot\|_{\mathbb{V}}};$$

  in general $\mathbb{V}_\infty \neq \mathbb{V}$
- criteria for the construction of $\mathbb{V}_N$ non obvious; should not require additional regularity of u
- convergence rate only meaningful in terms of dimension of $\mathbb{V}_N$ and only optimal for a good choice of spaces $\mathbb{V}_N$
  - any rate requires additional regularity of u above $\mathbb{V}$

Outline

Approximation of a Given Function

Piecewise Constant Approximation

approximate $u \in C^0([0,1])$ by a piecewise constant function U_N on a partition

$$0 = x_0 < x_1 < \cdots < x_N = 1$$

- for instance,

$$U_N(x) = u(x_n) \quad \text{for} \quad x \in I_n := (x_{n-1}, x_n], \qquad n = 1, \dots, N$$

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Convergence

uniform continuity of u implies

$$\lim_{N\to\infty} \|U_N - u\|_{L_\infty(0,1)} = 0$$

provided that

$$h_N := \max_{n=1,\ldots,N} |x_n - x_{n-1}| \to 0$$

Uniform and Adaptive Approximation

Uniform Approximation

choose x_n as

$$x_n = n/N, \qquad n = 1, \dots, N$$

- error bound for $u \in W^1_\infty(0,1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \le h_N \|u'\|_{L_\infty(0,1)} = N^{-1} \|u'\|_{L_\infty(0,1)}$$

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Adaptive Approximation

choose x_n s.th.

$$\int_{x_{n-1}}^{x_n} \left|u'(s)\right| \, ds = \frac{1}{N} \|u'\|_{L_1(0,1)}, \qquad n = 1, \dots, N$$

- error bound for $u \in W^1_1(0,1)$:

$$\|U_N - u\|_{L_\infty(0,1)} \le N^{-1} \|u'\|_{L_1(0,1)}$$

Remarks

Uniform Approximation

- quantification of the rate in terms of mesh-size and number of degrees of freedom is equivalent: $h_N = N^{-1}$

Adaptive Approximation

- quantification of the rate in terms of the mesh-size is impossible, only in terms of degrees of freedom; in general, $h_N \not\to 0$

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Adaptive Approximation

- quantification of the rate in terms of the mesh-size is impossible, only in terms of degrees of freedom; in general, $h_N \not\to 0$
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- choice of the grid points is independent of u and can be used for any function
- error bound needs stronger regularity $u \in W^1_\infty(0,1)$:
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Comparison of Uniform and Adaptive Approximation

Problem

given $u \in C^0([0,1])$ and TOL > 0 choose N minimal s.th.

$$\|U_N - u\|_{L_\infty(0,1)} \le \mathsf{TOL}$$

- task accomplished with minimal N s.th.

$$N_{\text{uniform}}^{-1} \|u'\|_{L_\infty(0,1)} \le \mathsf{TOL} \qquad \text{resp.} \qquad N_{\text{adaptive}}^{-1} \|u'\|_{L_1(0,1)} \le \mathsf{TOL}$$

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compare uniform and adaptive approximation for

$$u(x) := x^\alpha$$

in case of

$\alpha \ge 1$: $u \in W^1_\infty(0,1)$ (regular)

$0 < \alpha < 1$: $u \notin W^1_\infty(0,1)$ (singular) but $u \in W^1_1(0,1)$

Comparison for a Regular Solution

$\alpha \geq 1$:

$$\|u'\|_{L_\infty(0,1)} = \max_{x\in[0,1]} \alpha x^{\alpha-1} = \alpha, \qquad \|u'\|_{L_1(0,1)} = \int_0^1 \alpha x^{\alpha-1}\,dx = 1$$

- tolerances and number of nodes

$$N_{\text{uniform}} = \left\lceil \alpha \mathsf{TOL}^{-1} \right\rceil \quad \text{resp.} \quad N_{\text{adaptive}} = \left\lceil \mathsf{TOL}^{-1} \right\rceil$$

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Comparison for $\alpha = 10$

TOL	N_{uniform}	N_{adaptive}
1%	10^3	10^2
0.01%	10^5	10^4

- adaptive approximation more efficient but only by the factor $\alpha \geq 1$

Comparison for a Singular Solution

$\alpha < 1$: for uniform refinement we have

$$\|U_N - u\|_{L_\infty(0,1)} = |U_N(x_0) - u(x_0)| = |u(x_1)| = N^{-\alpha}$$

- tolerances and number of nodes

$$N_{\text{uniform}} = \left\lceil \mathsf{TOL}^{-1/\alpha} \right\rceil \quad \text{resp.} \quad N_{\text{adaptive}} = \left\lceil \mathsf{TOL}^{-1} \right\rceil$$

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Comparison for $\alpha = 0.1$

TOL	N_{uniform}	N_{adaptive}
1%	10^{20}	10^2
0.01%	10^{40}	10^4

- adaptive approximation independent of α!

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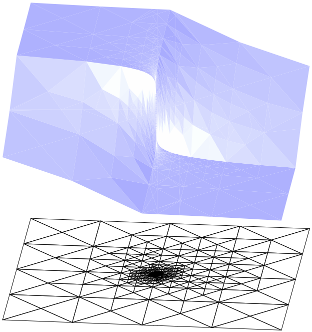
- adaptive approximation independent of α!
- uniform approximation with tolerance 1% on Hermit (1 PetaFlop) in Stuttgart will need more than 27h
- tolerance 0.01% trivial with adaptive approximation (< 1s on this laptop) but not feasible on any computer with uniform approximation

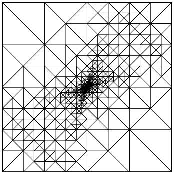
Connection to PDEs

typical singularities of solutions to PDEs are of the form $|x|^{\alpha}$, $\alpha \in (0, 1)$

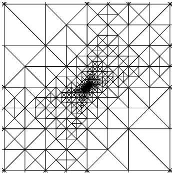
jumping coefficients



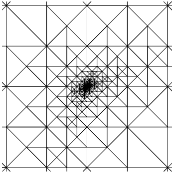
singularity $\approx |x|^{0.1}$



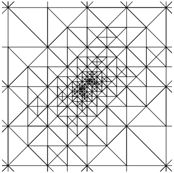
grid $<$ 2000 nodes



zoom: $(-10^{-3}, 10^{-3})^2$



zoom: $(-10^{-6}, 10^{-6})^2$

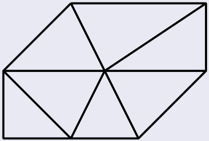


zoom: $(-10^{-9}, 10^{-9})^2$

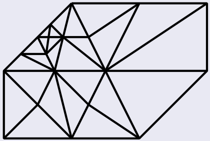
Outline

Adaptive Finite Element Discretization

- start with a coarse, conforming triangulation $\mathcal{G}_0$ of Ω
- let $\mathbb{G}$ be the set of all conforming refinements of $\mathcal{G}_0$ created by bisection



$\mathcal{G}_0$



$\mathcal{G} \in \mathbb{G}$

- let $\mathbb{V}(\mathcal{G}) \subset \mathbb{V}$ be a piecewise polynomial finite element space over $\mathcal{G}$.

Conforming Discretization with Adaptive Finite Elements

Adaptive Finite Element Discretization

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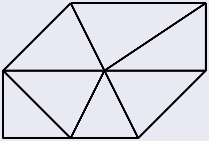
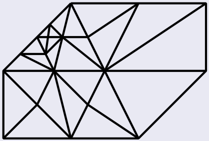


$\mathcal{G}_0$



$\mathcal{G} \in \mathbb{G}$

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bisection refinement of simplicial grids and polynomials is not essential but

Nesting of Spaces

if $\mathcal{G}_+$ is a refinement of $\mathcal{G}$ the finite element spaces are nested:

$$\mathcal{G} \le \mathcal{G}_+ \qquad \Longrightarrow \qquad \mathbb{V}(\mathcal{G}) \subset \mathbb{V}(\mathcal{G}_+)$$

Aim of Adaptive Finite Element Discretization

Aim of Adaptive Methods

given TOL > 0 find $\mathcal{G} \in \mathbb{G}$ s.th.

- $\|U_{\mathcal{G}} - u\|_{\mathbb{V}} \leq$ TOL
- $\#\mathcal{G}$ is minimal

constrained, discrete minimization problem . . .

Aim of Adaptive Finite Element Discretization

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- $\|U_\mathcal{G} - u\|_\mathbb{V} \le$ TOL
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constrained, discrete minimization problem ...
too hard to solve ...

Equidistribution Principle (Babuška, Rheinboldt, 1978)

optimal grid $\mathcal{G}$ satisfies

$$\|U_\mathcal{G} - u\|_{\mathbb{V}(E)} \equiv \text{const.} \qquad \forall E \in \mathcal{G}$$

Problems

- true solution u not known $\implies$ true error not known
- 1d error equidistribution does not extend to higher dimensions

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Adaptive Approaches

- replace the true error by an a posteriori estimator, i. e., a computable bound for the true error in terms of the discrete solution and data
- iteratively perform local refinement/coarsening of selected elements based on information of the estimator
- selection of elements subject for refinement/coarsening aims at equidistribution of the indicators

The Adaptive Loop for Stationary Problems

Standard Adaptive Iteration

iterate

SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE

to generate a sequence $\{\mathcal{G}_k, U_k\}_{k\geq 0}$ of grids and discrete solutions

- SOLVE: computes the Galerkin approximation $U_k \in \mathbb{V}_k = \mathbb{V}(\mathcal{G}_k)$
  - i. e., exact integration and exact linear algebra
- ESTIMATE: computes error indicators $\{\mathcal{E}_k(U_k, E)\}_{E\in\mathcal{G}_k}$
- MARK: selects elements in $\mathcal{G}_k$ for refinement
- REFINE: refines all marked elements and outputs a new grid

Error Estimation

Definition (Residual)

the residual $\mathcal{R}(U_\mathcal{G}) \in \mathbb{V}^*$ is an a posteriori quantity defined by

$$\langle \mathcal{R}(U_\mathcal{G}),\, v\rangle := \mathcal{B}[U_\mathcal{G},\, v] - \langle f,\, v\rangle \qquad \forall\, v \in \mathbb{V}$$

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$$\langle \mathcal{R}(U_\mathcal{G}), v \rangle := \mathcal{B}[U_\mathcal{G}, v] - \langle f, v \rangle = \mathcal{B}[U_\mathcal{G} - u, v] \qquad \forall\, v \in \mathbb{V}$$

Error and Residual

equivalence of $\|U_\mathcal{G} - u\|_\mathbb{V}$ to the non-computable quantity $\|\mathcal{R}(U_\mathcal{G})\|_{\mathbb{V}^*}$

$$c_\mathcal{B} \|U_\mathcal{G} - u\|_\mathbb{V} \le \|\mathcal{R}(U_\mathcal{G})\|_{\mathbb{V}^*} = \sup_{\|v\|_\mathbb{V}=1} \langle \mathcal{R}(U_\mathcal{G}), v \rangle \le \|\mathcal{B}\| \|U_\mathcal{G} - u\|_\mathbb{V}$$

- relies only on the continuous inf-sup condition and continuity of $\mathcal{B}$

Error Estimation

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Error Estimator

an estimator is a computable bound for the residual with

$$\|\mathcal{R}(U_\mathcal{G})\|_{\mathbb{V}^*} \lesssim \mathcal{E}_\mathcal{G}(U_\mathcal{G};\mathcal{G})$$
$$\mathcal{E}_\mathcal{G}(U_\mathcal{G};E) \lesssim \|\mathcal{R}(U_\mathcal{G})\|_{\mathbb{V}^*(\omega_E)} + \operatorname{osc}_\mathcal{G}(U_\mathcal{G};\omega_E), \qquad E \in \mathcal{G}$$

- constants solely depend on the shape regularity of $\mathcal{G}$

The Module ESTIMATE

Properties of an Estimator: Reliability and Efficiency

there are constants C_1, C_2 s.th. the Galerkin approximation $U_{\mathcal{G}}$ satisfies

$$C_1 \|U_{\mathcal{G}} - u\|_{\mathbb{V}}^2 \le \mathcal{E}_{\mathcal{G}}^2(U_{\mathcal{G}}; \mathcal{G}) := \sum_{E \in \mathcal{G}} \mathcal{E}_{\mathcal{G}}^2(U_{\mathcal{G}}; E)$$

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Example

- the indicators of the residual estimator for the Poisson problem read

$$\mathcal{E}_{\mathcal{G}}^2(U_{\mathcal{G}};E) := \|h_{\mathcal{G}}\,(-\Delta U_{\mathcal{G}} - f)\|_{L_2(E)}^2 + \|h_{\mathcal{G}}^{1/2}\,[\![\nabla U_{\mathcal{G}}]\!]\,\|_{L_2(\partial E\cap\Omega)}^2,$$

  where $h_{\mathcal{G}} \in L_\infty(\Omega)$ is the piecewise constant mesh-size function with

$$h_{\mathcal{G}|E} = |E|^{1/d} \approx \operatorname{diam}(E), \qquad E \in \mathcal{G}$$

- oscillation is here data oscillation and given by

$$\operatorname{osc}_{\mathcal{G}}^2(U_{\mathcal{G}};E) := \|h_{\mathcal{G}}(f_{\mathcal{G}} - f)\|_{L_2(E)}^2$$

The Module MARK

marking strategies select elements for refinement based on the indicators $\{\mathcal{E}_\mathcal{G}(U_\mathcal{G};E\}_{E\in\mathcal{G}}$; typical marking strategies are

Equidistribution Strategy

Parameter $\theta \in [0,1]$

$$\mathcal{E}_* = \theta\,\mathcal{E}_\mathcal{G}(U_\mathcal{G};\mathcal{G})/\#\mathcal{G}^{1/2}$$
$$\mathcal{M} = \{E \in \mathcal{G} \mid \mathcal{E}_\mathcal{G}(U_\mathcal{G};E) \geq \mathcal{E}_*\}$$

Maximum Strategy

Parameter $\nu \in [0,1]$

$$\mathcal{E}_* = \nu\,\max\{\mathcal{E}_\mathcal{G}(U_\mathcal{G};E) \mid E \in \mathcal{G}\}$$
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strategy of other type is

Dörfler Marking

Parameter $\theta \in (0,1]$

select any $\mathcal{M} \subset \mathcal{G}$ s.th.

$$\theta\mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}};\mathcal{G}) \le \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}};\mathcal{M})$$

- Dörfler Marking is also known as Fixed Fraction Marking, Bulk Criterion, Guaranteed Error Reduction Strategy

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Minimal Dörfler Marking

Parameter $\theta \in (0,1]$

select minimal $\mathcal{M} \subset \mathcal{G}$ s.th.

$$\theta \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}; \mathcal{G}) \leq \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}; \mathcal{M})$$

- Dörfler Marking is also known as Fixed Fraction Marking, Bulk Criterion, Guaranteed Error Reduction Strategy

Fundamental Questions and Problems

Basic Questions

1. does the sequence of adaptively generated solutions converge, i. e.,

$$\lim_{k\to\infty} \|U_k - u\|_{\mathbb{V}} = 0?$$

2. if yes, does the sequence converge with a rate, i. e.,

$$\|U_k - u\|_{\mathbb{V}} \lesssim |u|_s \left(\#\mathcal{G}_k - \#\mathcal{G}_0\right)^{-s}?$$

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Problems in the Analysis

1. all decision are taken at step k, no information about U_ℓ and $\mathcal{G}_\ell$ available for $\ell > k$
2. for scalar products $\mathcal{B}$ the quantity $|||U_k - u|||$ is monotone but not strictly monotone
3. the estimator $\mathcal{E}_k$ is not monotone, no relation between estimators of different iterations
4. U_k is a global object: refinement of a single element affects U_{k+1} everywhere; no quantification of these effects available

Outline

Basic Convergence Result (Morin, S., Veeser 2008, S. 2011)

Theorem (Convergence)

the adaptively generated sequence $\{U_k\}_{k\geq 0}$ converges in $\mathbb{V}$ to the solution u for the general class of inf-sup problems, provided

1. *the sequence of discrete spaces is nested and uniformly inf-sup stable;*
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Main Ideas of the Proof

1 uniform convergence of $\{h_k\}_{k\geq 0}$ to some $h_\infty \in L_\infty(\Omega)$

- $h_\infty \not\equiv 0$ in general

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1. uniform convergence of $\{h_k\}_{k\geq 0}$ to some $h_\infty \in L_\infty(\Omega)$
   - $h_\infty \not\equiv 0$ in general

2. strong convergence of $\{U_k\}_{k\geq 0}$ to the solution $u_\infty \in \mathbb{V}_\infty$ of the variational problem in

$$\mathbb{V}_\infty = \overline{\bigcup_{k\geq 0} \mathbb{V}_k}^{\|\cdot\|_{\mathbb{V}}} \subset \mathbb{V}$$

   - $\mathbb{V}_\infty = \mathbb{V}$ iff $h_\infty \equiv 0$

Basic Convergence Result (cont.)

Main Ideas of the Proof

$\boxed{3}$ convergence $U_k \to u_\infty$, $h_k \to h_\infty$, properties of $\mathcal{E}_k$ and MARK yield

$$\max\{\mathcal{E}_k(U_k; E) \mid E \in \mathcal{G}_k\} \to 0$$

Basic Convergence Result (cont.)

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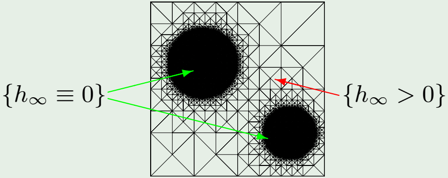
$\boxed{3}$ convergence $U_k \to u_\infty$, $h_k \to h_\infty$, properties of $\mathcal{E}_k$ and MARK yield

$$\max\{\mathcal{E}_k(U_k; E) \mid E \in \mathcal{G}_k\} \to 0$$

$\boxed{4}$ deduce $\mathcal{R}(u_\infty) = 0$ from

$$|\langle \mathcal{R}(U_k), v\rangle| = |\langle \mathcal{R}(U_k), v - I_k v\rangle| \lesssim \sum_{E \in \mathcal{G}_k} \mathcal{E}_k(U_k; E)\|v - I_k v\|_{\mathbb{V}(E)} \to 0$$

for smooth v by employing local density of the FE spaces in $\{h_\infty \equiv 0\}$, and convergence of the indicators in its complement $\{h_\infty > 0\}$. □



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plain convergence of adaptive finite elements is more or less settled

1. basic tool for a convergence result for the heat equation (Kreuzer, Möller, Schmidt, S. 2012)
2. transfer to control constrained optimal control problems (Kohls, Rösch, S. 2012)

generalizations seem to work for

1. other non-linear problems
   - needs strong convergence

     $$\|U_k - u_\infty\|_{\mathbb{V}} \to 0$$

     (for instance in strictly convex minimization problems)
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Drawback

arguments give no error reduction property, which is currently an essential tool to derive a convergence rate

- cannot be expected for this general problem class

Outline

Approximation Class

Quality of Best-Approximation

let $\mathbb{G}_N := \{\mathcal{G} \in \mathbb{G} \mid \#\mathcal{G} - \#\mathcal{G}_0 \le N\}$ and set

$$\sigma(N; u) := \min_{\mathcal{G} \in \mathbb{G}_N} \min_{V \in \mathbb{V}(\mathcal{G})} \|u - V\|$$

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for $s > 0$ define the approximation class

$$\mathcal{A}_s := \Big\{ u \mid |u|_s := \sup_{N>0} \big(N^s\, \sigma(N;u)\big) < \infty \Big\}$$

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Remarks

$u \in \mathcal{A}_s$ implies the existence of a sequence $\{\mathcal{G}_k\}_{k\ge 0}$ s.th.

$$\min_{V_k\in\mathbb{V}(\mathcal{G}_k)} \|V_k - u\| \lesssim |u|_s \left(\#\mathcal{G}_k - \#\mathcal{G}_0\right)^{-s} \to 0$$

- this is not the sequence produced by the adaptive method
- any rate $s > 0$ requires additional regularity of u above $\mathbb{V}$, e. g. H^1

Membership in $\mathcal{A}_{1/2}$ (Nochetto, S., Veeser, 2009)

Lemma (Constructive Approximation)

if $\mathbb{V} = H_0^1(\Omega)$, $d = 2$, *and* $u \in W_p^2(\Omega)$ *for some* $p > 1$ *then* $u \in \mathcal{A}_{1/2}$

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Remarks

1. constructive proof relying on
   - Lagrange interpolant $I_{\mathcal{G}} u$ being completely local
   - the strictly monotone and local error bound

$$\|I_{\mathcal{G}} u - u\|_E = \|\nabla(I_{\mathcal{G}} u - u)\|_{H^1(E)} \lesssim h_E^{2-2/p} \|D^2 u\|_{p;E}$$

2. proof based on Binev, Dahmen, DeVore, and Petrushev, who give a near characterization of $\mathcal{A}_s$ in terms of Besov spaces for Courant elements, 2002
3. rate only optimal for Courant elements in 2d
4. higher order elements and $d \geq 3$ need non-convex spaces ($p < 1$); compare with Gaspoz and Morin, 2011
5. proof requires a complexity result for refinement by bisection ...

Complexity of Refinement (Binev, Dahmen, DeVore 2004, Stevenson 2008)

Theorem (Complexity of Refinement by Bisection)

for a compatible labeling of refinement edges on $\mathcal{G}_0$ we have

$$\#\mathcal{G}_{k+1} - \#\mathcal{G}_0 \lesssim \sum_{\ell=0}^{k} \#\mathcal{M}_\ell \qquad \forall k \geq 0,$$

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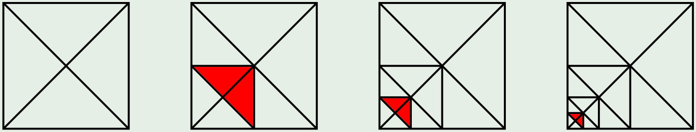
Remark

a bound

$$\#\mathcal{G}_{k+1} - \#\mathcal{G}_k \lesssim \#\mathcal{M}_k \qquad \forall k \geq 0$$

is not true:



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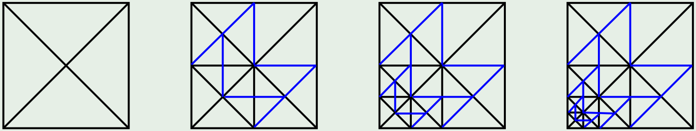

Assumptions for Optimality

Assumptions on AFEM

1. Poisson problem
2. the standard residual estimator $\mathcal{E}_{\mathcal{G}}$
3. *minimal* Dörfler Marking with $0 < \theta^2 < \theta_*^2 \approx C_2/C_1$

$$\theta \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}, \mathcal{G}) \le \mathcal{E}_{\mathcal{G}}(U_{\mathcal{G}}, \mathcal{M})$$

4. bisection and assumptions of the last theorem for $\mathcal{G}_0$

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Basic Property from Minimizing an Energy

for Ritz-projections $U \in \mathbb{V}(\mathcal{G})$ and $U_+ \in \mathbb{V}(\mathcal{G}_+)$ with $\mathbb{V}(\mathcal{G}) \subset \mathbb{V}(\mathcal{G}_+)$ we have the orthogonal error decomposition

$$|||U_+ - u|||^2 = |||U - u|||^2 - |||U - U_+|||^2$$

Contraction Property of AFEM (Cascón, Kreuzer, Nochetto, S. 2008)

Proposition (Contraction Property)

there are constants $\gamma > 0$ and $0 < \alpha < 1$ s.th.

$$|||U_{k+1} - u|||^2 + \gamma \mathcal{E}_{k+1}^2(U_{k+1}; \mathcal{G}_{k+1}) \le \alpha \Big(|||U_k - u|||^2 + \gamma \mathcal{E}_k^2(U_k; \mathcal{G}_k) \Big)$$

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Philosophy behind the Proof

combine the following two extreme cases:

1. $U_{k+1} = U_k$: then $\mathcal{E}_{k+1}^2(U_{k+1}; \mathcal{G}_{k+1}) \le \bar{\alpha} \mathcal{E}_k^2(U_k; \mathcal{G}_k)$ by the strict reduction of the local mesh-size and Dörfler Marking

$$\mathcal{E}_k^2(U_k; E) = \|h_k(-\Delta U_k - f)\|_{L_2(E)}^2 + \|h_k^{1/2} [\![\nabla U_k]\!]\|_{L_2(\partial E \cap \Omega)}^2$$

   - this case corresponds to large oscillation

2. zero oscillation: then $\|U_{k+1} - u\|^2 \le \bar{\alpha} \|U_k - u\|^2$ from the orthogonal error decomposition and Dörfler Marking
   - compensates even for a possible increase of $\mathcal{E}_k(U_k; \mathcal{G})$ $\square$

Optimal Rate (Stevenson 2006, Cascón, Kreuzer, Nochetto, S. 2008)

Theorem (Optimal Rate of AFEM)

if $u \in \mathcal{A}_s$ then the sequence of adaptively generated solutions U_k satisfies

$$|||U_k - u||| + \mathcal{E}_k(U_k; \mathcal{G}_k) \lesssim |u|_s \left(\#\mathcal{G}_k - \#\mathcal{G}_0\right)^{-s}$$

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Basic Steps of the Proof

$\boxed{1}$ localized upper bound for two Ritz approximations: a suitable error reduction yields the Dörfler property for the set of refined elements

$\boxed{2}$ minimal Dörfler Marking gives a connection of the adaptive mesh $\mathcal{G}_k$ with an optimal one

$\boxed{3}$ the connection to an optimal mesh yields an upper bound of $\#\mathcal{M}_k$ in terms of the actual error

$\boxed{4}$ proposition follows from complexity of REFINE and the contraction property $\square$

Remarks

Generalizations

1. extension to general scalar 2^{nd} order, linear, symmetric PDEs and higher order elements (Cascón, Kreuzer, Nochetto, S. 2008)
   - non-standard approximation class, depends also on oscillation
2. extension to other estimators being locally equivalent to the residual estimator (Kreuzer, S. 2011; Cascón, Nochetto 2012)

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Open Problems

1. other marking strategies are excluded
2. proof heavily relies on the orthogonal error decomposition; extension to non-symmetric and non-coercive $\mathcal{B}$ completely open
3. lack of a localized upper bound for two Ritz projections in case of $H(\mathrm{curl})$ or $H(\mathrm{div})$ problems
   - essential ingredient in the proof
4. . . .

Some References

Directly Related

1. Dörfler
2. Binev, Dahmen, DeVore
3. Cascón, Kreuzer, Nochetto, S.
4. Kreuzer, S.
5. Morin, Nochetto, S.
6. Morin, S., Veeser
7. Nochetto, S., Veeser
8. S.
9. Stevenson

Background

1. Bänsch, Morin, Nochetto
2. Carstensen
3. Cascón, Nochetto, S.
4. Chen, Feng
5. Chen, Holst, Xu
6. Carstensen, Hoppe
7. Diening, Kreuzer
8. Mekchay, Nochetto
9. Veeser
10. Veeser, S.
11. . . .

Thank you for your interest!