# On Singularity Formation Under Mean Curvature Flow

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Some of it uses ideas from work with Dimitra Antonopoulou and Georgia Karali

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# Mean Curvature Flow

The mean curvature flow of a hypersurface $M_0 \subset \mathbb{R}^{d+1}$ is a family of hypersurfaces $M_t \subset \mathbb{R}^{d+1}$ whose smooth immersions $\psi(\cdot, t) : N \to M_t \subset \mathbb{R}^{d+1}$ satisfy the partial differential equation

$$\partial_t \psi(z, t) = -h(\psi(z, t))$$

where $h(y)$ is the mean curvature vector of $M_t$ at a point $y \in M_t$, with the initial condition $\psi(z, 0) = \psi_0(z)$, an immersion of $M_0$.

# Applications and Connections

- Material Science (interface motion between different materials or different phases).
- Image recognition.
- Connection to the Ricci flow.
- Topological classification of surfaces and submanifolds.

# Some Key Works: Existence

- First mathematical treatment (using geometric measure theory): Brakke [1978];
- Short time existence: Brakke, Huisken, Evans and Spruck, Ilmanen, Ecker and Huisken [1991];
- Evans and Spruck, Chen, Giga and Goto [1991]: Weak solutions;

# Some Key Works: Singularities

The most interesting problem here is formation of singularities.

- Mu-Tao Wang, Kuo-Wei Lee and Yng-Ing Lee [2008]: Higher Co-dimensions;
- Huisken [1984]: Collapse of convex hypersurfaces;
- Grayson, Ecker, Huisken, M. Simon, Dziuk and Kawohl, Smoczyk, Altschuler, Angenent and Giga, Soner and Souganidis [1990-1995]: Neckpinching for periodic, rotationally symmetric;
- Huisken and Sinestrari [2007-2009]: MCF with surgery and topological classification of surfaces and submanifolds;
- B. White [2000], Colding and Minicozzi [2012]: Nature and Hausdorff dimension of the singular set.

# Classification of Singularities

There are three exact solutions of MCF:

- Collapsing Euclidean spheres with radii decreasing as $\sqrt{2d(t_* - t)}$;
- Collapsing Euclidean cylinders with radii decreasing as $\sqrt{2(d-1)(t_* - t)}$;
- Static planes.

Huisken conjectured that generic singularities are spheres and cylinders.

Colding and Minicozzi [2012] showed that the only 'linearly' stable self-similar solutions (shrinkers) are spheres.

# Stability of Shrinking Spheres

**Theorem.** (W. Kong-I.M.S.) Let a surface $M_0$ be close to $S^d$ in $H^s$, $s > \frac{d}{2} + 1$. Then $\exists t_* < \infty$, s.t. MCF has a solution $M_t$ for $0 \le t < t_*$ and

- $M_t \to z_*$, for some $z_*$, as $t \to t_*$;
- $M_t$ are defined by immersions of $S^d$,

$$X(\omega, t) = z(t) + R(\omega, t)\omega,$$

for some $z(t) \in \mathbb{R}^{d+1}$ and $R(t) \in H^s(\Omega)$, satisfying

$$z(t) = z_* + O(\tau^\alpha)$$

$$R(t) = \sqrt{2d\ \tau}\left(1 + O_{H^s}(\tau^\beta)\right),$$

with $\tau := t_* - t,\ \alpha := \frac{1}{2a}(d + \frac{1}{2} - \frac{1}{2d})$ and $\beta := \frac{1}{2a}(1 - \frac{1}{2d})$.

# Previous Results

- Colding and Minicozzi [2012] : 'linear' stablility of spherical shrinkers;
- Antonopoulou - Georgia Karali - IMS: asymptotics of volume preserving MCF near spheres

# Graphs over Cylinders

Our next result deals with initial conditions $M_0$, which are graphs over $d-$dimensional cylinders $C^d$ along the $x_{d+1}-$axis in $\mathbb{R}^{d+1}$,

$$X(\omega, x, t) = (u(\omega, x, t)\omega, x).$$

It combines two results, one with Zhou Gang on equivariant graphs (surfaces of revolution), i.e.

$$u(\omega, x, t) \text{ is independent of } \omega,$$

and one in general case with Zhou Gang and Dan Knopf, which is in preparation. For brevity, the initial conditions are formulated informally.

# Neckpinching

**Theorem.** (Zhou Gang-I. M. S., Zhou Gang-D. Knopf- IMS) Let $d \geq 2$ and $M_0$ be a surface close to a cylinder, $\mathcal{C}^d$,

$M_0$ has an arbitrary shalow waist and is even w.r.to the waist.

Then $M_t$ is defined by an immersion

$$X(\omega, x, t) = (u(\omega, x, t)\omega, x)$$

of $C^d$, where $(\omega, x) \in \mathcal{C}^d$ and $u(\omega, x, t)$ satisfies

(i) There exists a finite time $t^*$ such that $\inf u(\cdot, t) > 0$ for $t < t^*$ and $\lim_{t \to t^*} \inf u(\cdot, t) \to 0$;

(ii) If $u_0 \partial_x^2 u_0 \geq -1$ then there exists a function $u_*(\omega, x) > 0$ such that $u(\omega, x, t) \geq u_*(\omega, x)$ for $\mathbb{R} \backslash \{0\}$ and $t \leq t^*$.

# Dynamics of Scaling Parameter

**Theorem.** (Zhou Gang-I. M. S., Zhou Gang-D. Knopf- IMS)

(iii) There exist $C^1$ functions $\zeta(\omega, x, t)$, $\lambda(t)$, $a(t)$ and $b(t)$ such that

$$u(\omega, x, t) = \lambda(t)[(\frac{2(d-1) + b(t)y^2}{a(t)})^{\frac{1}{2}} + \zeta(\omega, y, t)]$$

with $y := x/\lambda(t)$, $\|\langle y\rangle^{-m}\partial_y^n \zeta(\omega, x, t)\|_\infty \leq cb^2(t)$, $m+n = 3$,

(iv) The parameters $\lambda(t)$, $a(t)$ and $b(t)$ satisfy the estimates

$$\begin{aligned} \lambda(t) &= (t^* - t)^{\frac{1}{2}}(1 + o(1)); \\ b(t) &= -\frac{d-1}{ln|t^*-t|}(1 + O(\frac{1}{|ln|t^*-t|}|^{3/4})); \\ a(t) &= 1 + \frac{1}{ln|t^*-t|}(1 + O(\frac{1}{ln|t^*-t|})). \end{aligned}$$

# Comparison with Previous Results

A result similar to (ii) ( axi-symmetric surfaces) but for a different set of initial conditions was proven by H.M.Soner and P.E.Souganidis.

The previous result closest to ours is that by S. Angenent and D. Knopf on the axi-symmetric neckpinching for the Ricci flow.

Some ideas of the proof are close to those of Bricmont and Kupiainen on NLH.

All works mentioned above deal with *surfaces of revolution* of barbell shapes (*far from cylinders*) which are either compact (Dirichlet b.c.) or periodic (Neumann b.c.).

These works rely on parabolic maximum principle going back to Hamilton and monotonicity formulae for an entropy functional $\int_{M_t}$ backward heat kernel$(x,t)d\mu_t$, due to Huisken and Giga and Kohn.

# Key Steps in Proof

Passing to collapse variables;

Collar lemma;

Estimates of the linear evolution;

Bootstrap.

## Collapse Variables

We outline key steps in proof in the *axi-symmetric* case. We pass from the original variables $x$ and $t$ to the collapse variables

$$y := \lambda^{-1}(t)(x - x_0(t)) \text{ and } \tau := \int_0^t \lambda^{-2}(s)\, ds.$$

*Important point*: we do not fix $\lambda(t)$ and $x_0(t)$ but consider them as free parameters to be found from (MCF).
For a profile $u(x, t)$ we define the new unknown function

$$v(y, \tau) := \lambda^{-1}(t)u(x, t)$$

with $y := \lambda^{-1}(t)x$ and $\tau := \int_0^t \lambda^{-2}(s)ds$. The function $v$ satisfies the equation

$$\partial_\tau v = (\frac{1}{1 + (\partial_y v)^2}\partial_y^2 - ay\partial_y + a)v - \frac{d-1}{v}, \text{ with } a := -\lambda\partial_t\lambda.$$

# Adiabatic Solutions

The rescaled MC equation has the following cylindrical, static (i.e. $y$ and $\tau$-independent) solution

$$V_a := (\frac{d-1}{a})^{\frac{1}{2}} \text{ and } a \text{ is constant } \iff u_{cyl}(t).$$

A larger family of approximate solutions: ignore $\partial_\tau v$ and $\partial_y^2 v$ in the equation for $v \implies$

$$ay\partial_y v - av + \frac{d-1}{v} = 0$$

(adiabatic or slowly varying approximation). This equation has the general solution

$$V_{ab} := (\frac{2(d-1) + by^2}{2a})^{\frac{1}{2}}$$

with $b \in \mathbb{R}$. In what follows we take $b \geq 0$ so that $V_{ab}$ is smooth. Note that $V_{a0} = V_a$.

# Collar Lemma

We introduce the manifold of adiabatic solutions
$M_{neck} := \{V_{ab} \mid a, b \in \mathbb{R}^+, b \le \epsilon\}$.

### Lemma

*There exist a small neigbourhood $\mathcal{U}_{path}$ of $M_{neck}$ in $C^1([0,T], \langle y\rangle^3 L^\infty)$ and a unique $C^1$ map $g : \mathcal{U}_{path} \to C^1([0,\infty),\mathbb{R}) \times C^1([0,\infty),\mathbb{R})$, such that*

$$v(y,\tau) = V_{g(v)(\tau)}(y) + \phi(y,\tau),$$

*with $g(u)(\tau) = (a(\tau), b(\tau))$, and*

$$\phi(\cdot,\tau) \perp 1,\ a(\tau)y^2 - 1 \ in\ L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy).$$

Similarly, for the initial conditions $v_0(y) := \lambda_0^{-1} u_0(\lambda_0 y)$ .
The vectors 1 and $(1 - ay^2)$ which are almost tangent vectors to the manifold, $M_{neck}$, provided $b$ is sufficiently small.

# Lyapunov-Schmidt Splitting (Effective Equations)

Substitute $\quad v(y,\tau) = V_{a(\tau),b(\tau)}(y) + \phi(y,\tau)$
into (MCF) to obtain

$$\partial_\tau \phi = -L_{ab}\phi + F_{ab} + N_{ab}(\phi)$$

where $L_{ab}$ is the linear operator given by

$$L_{ab} := -\partial_y^2 + ay\partial_y - 2a + V_{ab}(y)$$

$F_{ab} \approx$ a sum of generators of broken symmetries (the source term) and $N_{ab}(\phi)$ is a nonlinearity. Remember that

$$\phi(\cdot,\tau) \perp 1,\ a(\tau)y^2 - 1 \text{ in } L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2}y^2}dy).$$

Project the above equation on $1,\ a(\tau)y^2 - 1 \implies$ the *equations for the parameters* $a,\ b \implies$ *need to estimate* $\phi$.

# Estimating $\phi$

Let $U(\tau, \sigma)$ be the propagator generated by $-L_a$. By Duhamel principle we rewrite the differential equation for $\phi(y, \tau)$ as

$$\phi(\tau) = U(\tau, 0)\phi(0) + \int_0^\tau U(\tau, \sigma)(F + N)(\sigma) d\sigma.$$

Using the *key propagation estimate* ( $\tau \geq \sigma \geq 0$)

$$\|\langle z \rangle^{-3} U(\tau, \sigma) g\|_\infty \lesssim e^{-c(\tau-\sigma)} \|\langle z \rangle^{-3} g\|_\infty,$$

where $g \perp 1,\ a(\tau)y^2 - 1$ in $L^2(\mathbb{R}, e^{-\frac{a(\tau)}{2} y^2} dy)$, we obtain

$$M(\tau) \leq M(0) + b^{\frac{1}{2}}(0) P(M(\tau)),$$

where

$$M(\tau) := \max_{\sigma \leq \tau} b^{-\frac{m+n+1}{2}}(\sigma) \|\langle y \rangle^{-m} \partial_y^n \phi(\cdot, \sigma)\|_\infty.$$

# Estimating the Linear Propagator. I

Denote the integral kernel of $U(\tau, \sigma)$ by $K(x, y)$. We have the representation

$$K(x, y) = K_0(x, y)\langle e^V \rangle(x, y),$$

where $K_0(x, y)$ is the integral kernel of the operator $e^{-(\tau-\sigma)L_{a0}}$, $L_{a0} := -\partial_y^2 + ay\partial_y - 2a$, $V := L_{ab} - L_{a0}$ and

$$\langle e^V \rangle(x, y) = \int e^{\int_\sigma^\tau V(\omega(s)+\omega_0(s),s)ds} d\mu(\omega).$$

Here $d\mu(\omega)$ is a harmonic oscillator (Ornstein-Uhlenbeck) probability measure on the continuous paths $\omega : [\sigma, \tau] \to \mathbb{R}$ with the boundary condition $\omega(\sigma) = \omega(\tau) = 0$ and

$$(-\partial_s^2 + a^2)\omega_0 = 0 \text{ with } \omega(\sigma) = y \text{ and } \omega(\tau) = x.$$

# Estimating the Linear Propagator. II

To estimate $U(x, y)$ for $e^{a(\tau-\sigma)} \leq b^{-1/32}(\tau)$ we use the explicit formula

$$K_0(x, y) = 4\pi(1 - e^{-2ar})^{-\frac{1}{2}}\sqrt{a}e^{2ar}e^{-a\frac{(x-e^{-ar}y)^2}{2(1-e^{-2ar})}},$$

where $r := \tau - \sigma$, and the bound

$$|\partial_y \langle e^V \rangle(x, y)| \leq b^{\frac{1}{2}} r,$$

which follows from the definition of $\langle e^V \rangle$ and the properties

$$V(y, \tau) \geq 0 \text{ and } |\partial_y V(y, \tau)| \lesssim b^{\frac{1}{2}}(\tau).$$

Then we iterate using the semi-group property $\Rightarrow$ estimate of the remainder $\phi$.

# Open Problems

Ricci flow;

Anisotropic mean curvature flow
(Euclidean spheres $\Rightarrow$ Wulff shapes);

Kähler surfaces;

Going through the singularity.

*Thank-you for your attention.*

# *Comparison with Yang-Mills and Wave Maps Equations*

Compare the dynamics for the scaling parameter $\lambda(t)$ for (MCF) and the critical Yang-Mills equation

$$\lambda\ddot{\lambda} = \frac{3}{4}\dot{\lambda}^4,$$

which gives

$$\lambda \approx \sqrt{\frac{2}{3}}\frac{t_* - t}{\sqrt{-\ln(t_* - t)}}.$$

and the critical wave map equation

$$\dot{\lambda}^2 = \lambda\ddot{\lambda}\ln\frac{a}{\lambda\ddot{\lambda}}\ ,\ \ a = 0.122.$$

# *MCF for Surfaces of Revolution*

If $d \geq 2$ and $M_0$ is a surface of revolution around the axis $x = x_{d+1}$, given by a map $r = u_0(x)$ where $r = (\sum_{j=1}^{d} x_j^2)^{\frac{1}{2}}$, then $M_t$ is also a surface of revolution and, as long as it is smooth. $M_t$ is defined by the map $r = u(x,t)$, satisfying the PDE

$$\begin{aligned} \partial_t u &= \frac{\partial_x^2 u}{1+(\partial_x u)^2} - \frac{d-1}{u} \qquad \text{(MCF)} \\ u(x,0) &= u_0(x). \end{aligned}$$

Our goal is to describe the phenomenon of collapse of such surfaces.
We say $u(x,t)$ collapses at time $t^*$ if $\|\frac{1}{u(\cdot,t)}\|_\infty < \infty$ for $t < t^*$ and $\|\frac{1}{u(\cdot,t)}\|_\infty \to \infty$ as $t \to t^*$.

# *Precise restrictions on the initial conditions*

$u_0(x)$ is even and satisfy for $(m,n)=(3,0)$, $(\frac{11}{10},0)$, $(1,2)$ and $(2,1)$ the estimates

$$\| u_0(x) - (\tfrac{2(d-1)+\varepsilon_0 x^2}{2\varsigma_0})^{\frac{1}{2}} \|_{m,n} \leq C\varepsilon_0^{\frac{m+n+1}{2}},$$

$$u_0(x) \geq \frac{1}{\sqrt{2\varsigma_0 + \frac{\varepsilon_0}{d-1}}} g(\sqrt{2\varsigma_0 + \tfrac{\varepsilon_0}{d-1}}x, \tfrac{\varepsilon_0}{2\varsigma_0}),$$

$$\langle x\rangle^{-1} u_0 \in L^\infty,\ \partial_x u_0 \in L^\infty,\ |\partial_x u_0 u_0^{-\frac{1}{2}}| \leq \kappa_0 \varepsilon_0^{\frac{1}{2}},\ |\partial_x^n u_0| \leq \kappa_0 \varepsilon_0^{n/2},\ n=2,3$$

for some $C$, $\kappa_0 \geq 2$, and $\frac{1}{2} \leq \varsigma_0 \leq 2$.

# *Comparison with Ricci Flow*

A compact surface of revolution can be given by the metric

$$g = \varphi^2 dx^2 + u^2 g_{can} \text{ on } (-1, 1) \times S^d,$$

where $g_{can}$ is the round metric of radius 1 on $S^d$ (a $SO(d+1)$-invariant metric on $S^{d+1}$). ($u(x, t) > 0$ may be regarded as the radius of the hypersurface $\{x\} \times S^d$.)
In this representation the Ricci flow for surfaces of revolution is

$$\partial_t u \quad = \quad \partial_s^2 u - (d-1)\frac{1-(\partial_s u)^2}{u},$$

where $s(x) := \int_0^x \varphi(y) dy$.
(MCF can also be written as a flow of the metric $g$.)

# New elements

Non-compact, non-periodic surfaces are allowed;

Initial conditions include in particular surfaces whose mean curvature changes sign and which might have many necks;

The subleading term in the asymptotic is determined and the remainder is estimated.