Minimal graphs and Bernstein's Theorem

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Outline

Minimal surfaces

- It is well known that given two points p and q in $\mathbb{R}^3$, the line segment minimizes the length among all curves joining p and q.

The 2-dimensional analog of this problem can be stated in the following way

- Replacing the pair of points by a "nice" closed curve C in $\mathbb{R}^3$, find a surface with boundary C which minimizes the area among all surfaces with boundary C.

It turns out that the solutions to this problem are necessarily minimal surfaces.

Definition

A surface in $\mathbb{R}^3$ is called *minimal* if its mean curvature H vanishes identically.

The minimal surface equation

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One can construct a huge class of surfaces in $\mathbb{R}^3$ just by considering graphs of smooth functions. The graph G_f of a smooth function $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is the surface

$$G_f = \{(x, y, f(x, y)) | (x, y) \in D\}.$$

Lemma

The graph G_f is a minimal surface if and only f satisfies

$$(1 + f_y^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0.$$

This is the minimal surface equation, which is actually the Euler-Lagrange equation for the area integral.

Bernstein's Theorem for surfaces

Of the considerable literature devoted to solutions of the the minimal surface equation, the part that has undoubtedly excited most interest is that related to Bernstein's Theorem.

Theorem (Bernstein, 1927)

The only entire minimal graphs in $\mathbb{R}^3$ are planes. In other words the only entire solutions to minimal surface equation are the linear ones.

Actually, Bernstein obtained the above result as an application of another deep result known as Bernstein's geometric theorem.

Bernstein's geometric theorem

Theorem (Bernstein's geometric theorem)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an entire smooth function. If the graph G_f of f has Gaussian curvature $K \leq 0$ and $K < 0$ at some point, then f cannot be bounded.

As an application of this theorem, Bernstein obtained the following very general Liouville theorem.

Theorem (Bernstein's geometric theorem)

Every bounded entire solution $u(x, y)$ to the equation

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = 0,$$

is constant, where A, B, C are arbitrary functions of x, y as well as u and its derivatives, such that $AC - B^2 > 0$.

An immediate consequence is that every bounded entire solution of the minimal surface equation is constant. However Bernstein observed that if $f(x,y)$ satisfies the the minimal surface equation then the functions

$$u = \tan^{-1} f_x \text{ and } v = \tan^{-1} f_y$$

satisfy the equation

$$(1+f_y^2)u_{xx} - 2f_x f_y u_{xy} + (1+f_x^2)u_{yy} = 0.$$

It follows that f_x and f_y are both constant. Thus Bernstein derived the far stronger and much more unexpected result that any entire solution (with no boundeness or growth assumptions) must be linear.

Bernstein's Theorem for higher dimension

So the theorem of S. Bernstein [2] states that the only entire minimal graphs in the Euclidean space $\mathbb{R}^3$ are planes. Equivalently, if $u : \mathbb{R}^2 \to \mathbb{R}$ is an entire smooth solution of the differential equation

$$\operatorname{div}\left(\frac{\operatorname{grad} u}{\sqrt{1 + |\operatorname{grad} u|^2}}\right) = 0,$$

then u is an affine function. It was conjectured for a long time that the theorem of Bernstein holds in any dimension for $u : \mathbb{R}^n \to \mathbb{R}$. However, for $n = 3$, its validity was proved by E. De Giorgi [1], for $n = 4$ by F. Almgren [1] and for $n = 5, 6, 7$ by J. Simons [4]. It was a big surprise when E. Bombieri, E. De Giorgi and E. Giusti [3] proved that, for $n \geq 8$, there are entire solutions of the minimal surface equation other than the affine ones.

Bernstein's Theorem for higher dimension

We are interested in Bernstein's type theorems for higher codimension. In particular we deal with minimal surfaces M^2 which arise as graphs over vector valued maps $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1, f_2)$, that is

$$M^2 = G_f := \left\{ (x, y, f_1(x, y), f_2(x, y)) \in \mathbb{R}^4 : (x, y) \in \mathbb{R}^2 \right\}.$$

There are plenty of complete minimal graphs in $\mathbb{R}^4$, other than the planes. More precisely, if $f : \mathbb{C} \to \mathbb{C}$ is any entire holomorphic or anti-holomorphic function, then the graph G_f of f in $\mathbb{C}^2 = \mathbb{R}^4$ is a minimal surface and is called a *complex analytic curve*.

It should be noticed that R. Osserman in [5], has constructed examples of complete minimal two dimensional graphs in $\mathbb{R}^4$, which are not complex analytic curves with respect to any orthogonal complex structure on $\mathbb{R}^4$. For instance, the graph G_f over the map $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by

$$f(x,y) = \frac{1}{2}\left(e^x - 3e^{-x}\right)\left(\cos\frac{y}{2}, -\sin\frac{y}{2}\right), \quad (x,y) \in \mathbb{R}^2,$$

is such an example. It is worth noticing that the *Jacobian* $J_f := \det(df)$ of f given by $J_f = -(e^{2x} - 9e^{-2x})/8$ takes every real value.

The problem that we deal with, is to find under which geometric conditions the minimal graph of an entire vector valued map $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f = (f_1, f_2)$, is a complex analytic curve. The first result was obtained by S.S. Chern and R. Osserman [4], where they proved that if the differential df of f is bounded, then G_f must be a plane. A few years later, L. Simon [3] obtained a much more general result by proving that if one of f_1, f_2 has bounded gradient, then f is affine. Later on, R. Schoen [2] obtained a Bernstein type result by imposing the assumption that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Moreover, L. Ni [4] has derived a result of Bernstein type under the assumption that f is an *area-preserving map*, that is the Jacobian J_f satisfies $J_f = 1$. This result was generalized by the authors in a previous paper [3], just by assuming that J_f is bounded.

We prove the following result of Bernstein type, from which known results due to R. Schoen [2], L. Fu [2], L. Ni [4], follow as consequences.

Theorem (Th. Hasanis, A. Savas-Halilaj and —)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an entire smooth vector valued map such that its graph G_f is a minimal surface in $\mathbb{R}^4$. Assume that G_f is not a plane. Then, the graph G_f over f is a complex analytic curve if and only if the Jacobian J_f of f does not take every real value. In particular, if G_f is a complex analytic curve, then the Jacobian J_f takes every real value in $(0, +\infty)$ or in $[0, +\infty)$ (resp. $(-\infty, 0)$ or in $(-\infty, 0]$), if f is holomorphic (resp. anti-holomorphic).

Proof

Let $f = (f_1, f_2)$ be an entire solution of the minimal surface equation (2.1). Then its graph,

$$G_f = \left\{ (x, y, f_1(x,y), f_2(x,y)) \in \mathbb{R}^4 : (x,y) \in \mathbb{R}^2 \right\},$$

is a minimal surface. According to a result due to Osserman, we can introduce global isothermal parameters, via a non-singular transformation

$$x = u, \quad y = au + bv,$$

where a, b are real constants with $b > 0$. Now, the minimal surface G_f is parametrized by the map

$$X(u,v) = (u, au + bv, \varphi(u,v), \psi(u,v)),$$

where

$$\varphi(u,v) := f_1(u, au + bv) \quad \text{and} \quad \psi(u,v) := f_2(u, au + bv).$$

Set $\Phi = (\varphi, \psi)$. Because of the relation

$$\frac{\partial(\varphi, \psi)}{\partial(u, v)} = \frac{\partial(f_1, f_2)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)}$$

for the Jacobians, we obtain

$$J_\Phi = bJ_f.$$

Since (u, v) are isothermal parameters and taking into account that G_f is minimal, it follows that the functions φ and ψ are harmonic, that is

$$\varphi_{uu} + \varphi_{vv} = 0 = \psi_{uu} + \psi_{vv}.$$

Then, the complex valued functions $\phi_k : \mathbb{C} \to \mathbb{C}$, $k = 1, 2, 3, 4$, given by

$$\begin{cases} \phi_1 = 1, & \phi_2 = a - ib, \\ \phi_3 = \varphi_u - i\varphi_v, & \phi_4 = \psi_u - i\psi_v \end{cases} \tag{1}$$

are holomorphic and satisfy

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 0. \tag{2}$$

Proof of Theorem

Assume that the graph G_f of $f(x,y) = (f_1(x,y), f_2(x,y))$, $(x,y) \in \mathbb{R}^2$, is a minimal surface which is not a plane.
Suppose now that J_f does not take every real value. We shall prove that G_f is a complex analytic curve.
On account of (1), equation (2) can be written equivalently in the form

$$(\phi_3 - i\phi_4)(\phi_3 + i\phi_4) = -d, \tag{3}$$

where $d = 1 + (a - ib)^2$. We claim that $d = 0$. Assume to the contrary that $d \neq 0$.
By virtue of (3), we see that $\phi_3 - i\phi_4$, $\phi_3 + i\phi_4$ are entire nowhere vanishing holomorphic functions. Define the complex valued function $h : \mathbb{C} \to \mathbb{C}$, by

$$h = \phi_3 - i\phi_4. \tag{4}$$

We point out that h is holomorphic, non-constant and nowhere vanishing. Combining (3) with (4), we get

$$\phi_3 = \frac{1}{2}\left(h - \frac{d}{h}\right) \quad \text{and} \quad \phi_4 = \frac{i}{2}\left(h + \frac{d}{h}\right). \tag{5}$$

Bearing in mind (1), it follows that the imaginary part of $\phi_3\overline{\phi}_4$ is given by

$$\mathrm{Im}(\phi_3\overline{\phi}_4) = \varphi_u\psi_v - \varphi_v\psi_u = J_\Phi.$$

On the other hand, from (5) we get

$$\mathrm{Im}(\phi_3\overline{\phi}_4) = \frac{1}{4}\left(-|h|^2 + \frac{|d|^2}{|h|^2}\right).$$

Thus, taking into account the relation $J_\Phi = bJ_f$, we have

$$J_f = \frac{1}{4b}\left(-|h|^2 + \frac{|d|^2}{|h|^2}\right).$$

Since h is an entire and non-constant holomorphic function, by Picard's Theorem, there are sequences $\{z_n\}_{n\in\mathbb{N}}$ and $\{w_n\}_{n\in\mathbb{N}}$ of complex numbers such that $|h(z_n)| \to \infty$ and $|h(w_n)| \to 0$. Consequently, $J_f(z_n) \to -\infty$ and $J_f(w_n) \to \infty$. Thus, $J_\Phi(\mathbb{R}^2) = \mathbb{R}$, which contradicts to our assumptions.
Thus $d = 0$, $a = 0$, $b = 1$ and consequently (x, y) are isothermal parameters. Furthermore, from (3) we obtain $\phi_3 = \pm i\phi_4$, or equivalently,

$$\frac{\partial f_1}{\partial x} - i\frac{\partial f_1}{\partial y} = \pm i\left(\frac{\partial f_2}{\partial x} - i\frac{\partial f_2}{\partial y}\right). \tag{6}$$

From (6) we readily deduce that $f = f_1 + if_2$ is holomorphic or anti-holomorphic. Therefore, G_f must be a complex analytic curve.

Conversely, assume that G_f is a complex analytic curve which is not a plane. Then, the complex valued function $f = f_1 + if_2$ is holomorphic or anti-holomorphic. We introduce the complex variable $z = x + iy$. An easy computation shows that

$$J_f = |f_z|^2 - |f_{\bar{z}}|^2 .$$

Hence, $J_f \geq 0$ if f is holomorphic and $J_f \leq 0$ if f is anti-holomorphic. In either case, J_f does not take every real value.
If f is holomorphic, then,

$$J_f = |f_z|^2 .$$

By the fact that f is holomorphic, we obtain that f_z is an entire holomorphic function. Moreover f_z cannot be constant, since otherwise f is affine and G_f a plane. Consequently, appealing to Picard's Theorem, the range of f_z is the whole complex plane $\mathbb{C}$, or the plane minus a single point. Thus, the range of J_f must be $(0, +\infty)$ or $[0, +\infty)$.
In the case where f is anti-holomorphic, we have

$$J_f = -\left|f_{\bar{z}}\right|^2$$

and $f_{\bar{z}}$ is an entire anti-holomorphic function. Arguing as above, we deduce that the range of J_f must be $(-\infty, 0)$ or $(-\infty, 0]$.

Applications

The following result due to R. Schoen [2] is a consequence of Theorem 1.1.

Corollary

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an entire solution of the minimal surface equation. If f is a diffeomorphism, then f is an affine map.

Proof.

Since f is a diffeomorphism, it follows that $J_f > 0$ if f is orientation preserving, or $J_f < 0$ if f is orientation reversing. Then, by Theorem 1.1, f must be holomorphic or anti-holomorphic. Thus, f is an entire conformal diffeomorphism. By a classical theorem of Complex Analysis [1, p. 388], f must be an affine map. $\square$

Finally, we provide an alternative proof of Jörgens Theorem.

Theorem (Jörgens' Theorem)

The only entire solutions $f : \mathbb{R}^2 \to \mathbb{R}$ of the Monge-Ampere equation $f_{xx} f_{yy} - f_{xy}^2 = 1$ are the quadratic polynomials.

Proof of Jörgens' Theorem.

Obviously $f_{xx} + f_{yy} \neq 0$ everywhere on $\mathbb{R}^2$. By virtue of classical regularity results for elliptic equations, $f(x, y)$ actually is a real analytic function. We consider the function $\Theta : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\Theta = \frac{f_{xx} f_{yy} - f_{xy}^2 - 1}{f_{xx} + f_{yy}}.$$

The function Θ, thanks to our assumption, is identically zero, so $\Theta_x = \Theta_y = 0$. On the other hand, one can readily verify that the equations $\Theta_x = \Theta_y = 0$ are equivalent to the minimal surface equation (2.4) for the function $g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(x, y) = (f_x(x, y), f_y(x, y))$. Moreover, we have $J_g = 1$. So, the graph G_g of g is a plane and the result is immediate. $\square$

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